Section 4.2
Radian, Arc Length, and Area of a Sector

An angle is formed by two rays that have a common endpoint (vertex). One ray is the initial side and the other is the terminal side. We typically will draw angles in the coordinate plane with the initial side along the positive x axis.

We measure angles in two different ways, both of which rely on the idea of a complete revolution in a circle. The first is degree measure. In this system of angle measure one complete revolution is $360^\circ$. The second method is called radian measure. One complete revolution is $2\pi$. The problems in this section are worked in radians. Radian is a unit free measurement.

The Radian Measure of an Angle

Place the vertex of the angle at the center of a circle of radius $r$. Let $s$ denote the length of the arc intercepted by the angle. The radian measure $\theta$ of the angle is the ratio of the arc length $s$ to the radius $r$. In symbols, $\theta = \frac{s}{r}$. In this definition it is assumed that $s$ and $r$ have the same linear units.

Example 1: A central angle, $\theta = \frac{\pi}{2}$, in a circle intercepts an arc of length $\frac{12\pi}{5}$ m. What is the radius of the circle?

$\text{Recall: } \theta = \frac{s}{r}$
Relationship between Degrees and Radians

How can we obtain a relationship between degrees and radians? We compare the number of degrees and the number of radians in one complete rotation in a circle. We know that $360^\circ$ is all the way around a circle. The length of the intercepted arc is equal to the circumference of the circle. Therefore, the radian measure of this central angle is the circumference of the circle divided by the circle’s radius, $r$. The circumference of a circle of a radius $r$ is $2\pi r$.

We use the formula for radian measure to find the radian measure of the $360^\circ$ angle.

$$\theta = \frac{s}{r} = \frac{\text{the circle's circumference}}{r} = \frac{2\pi r}{r} = 2\pi$$

So, $360^\circ = 2\pi$ radians.

Dividing both sides by 2, we get $180^\circ = \pi$ radians. Dividing this last equation by $180^\circ$ or $\pi$ gives the conversion rules that follow.

Conversion between Degrees and Radians

Using the fact that $\pi$ radians $= 180^\circ$,

1. To convert degrees to radians, multiply degrees by $\frac{\pi \text{ radians}}{180^\circ}$.
2. To convert radians to degrees, multiply radians by $\frac{180^\circ}{\pi \text{ radians}}$.

Note: The unit you are converting to appears in the numerator of the conversion factor.

Example 2: Convert the given angle to radians.

$-135^\circ$

Example 3: Convert the given angle to degrees.

$\frac{7\pi}{9}$
Common Angles (Memorize these!)

<table>
<thead>
<tr>
<th>Angle (°)</th>
<th>Equivalent (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>360°</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>180°</td>
<td>$\pi$</td>
</tr>
<tr>
<td>90°</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$</td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$</td>
</tr>
</tbody>
</table>

Sector Area Formula

In a circle of radius $r$, the area $A$ of a sector with central angle of radian measure $\theta$ is given by

$$A = \frac{1}{2} r^2 \theta.$$

Example 4: Given a circle the area of sector is $\frac{\pi}{3}$ in$^2$ and the central angle is $\frac{\pi}{6}$. Find the radius.

Example 5: Find the perimeter of a sector with central angle 60° and radius 3 m.
Linear and Angular Velocity (Speed)

Consider a merry-go-round

The ride travels in a circular motion.

Some of the horses are right along the edge of the merry-go-round, and some are closer to the center. If you are on one of the horses at the edge, you will travel farther than someone who is on a horse near the center. But the length of time that both people will be on the ride is the same. If you were on the edge, not only did you travel farther, you also traveled faster. However, everyone on the merry-go-round travels through the same number of degrees (or radians).

There are two quantities we can measure from this, angular velocity and linear velocity.

The angular velocity of a point on a rotating object is the number of degrees (or radians or revolutions) per unit of time through with the point turns. This will be the same for all points on the rotating object. Units will be in the form radians/time, degrees/time or revolutions/time. We let the Greek letter \( \omega \) (omega) represent angular velocity. Using the definition above, \( \omega = \frac{\theta}{t} \).

\[ 2\pi = 360^\circ = 1 \text{ revolution} \]

The linear velocity of a point on the rotating object is the distance per unit of time that the point travels along its circular path. This distance will depend on how far the point is from the axis of rotation (for example, the center of the merry-go-round). Units will be in the form km/hr, m/s, mph, etc.

We can establish a relationship between the two kinds of speed by multiplying angular velocity by the radius \( r \). We obtain linear speed in terms of angular speed: \( r \frac{\theta}{t} \).

Example 6: A car has wheels with a 10 inch radius. If each wheel’s rate of turn is 4 revolutions per second,

a. Find the angular speed in units of radians/second.

b. How fast (linear speed) is the car moving in units of inches/second?