Taylor Polynomials

General procedure for finding an infinite polynomial series that converges to a "non-polynomial" function.

Review Factorials -

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

where $0! = 1$

**Taylor Polynomial at 0**

The $n$th degree Taylor polynomial for $f$ at $x = 0$ is

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

provided $f$ has $n$ derivatives at 0.

Example $\quad f(x) = e^x \qquad P_4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ | $\frac{f^{(k)}(0)}{k!}$ |
|---|---|---|---|
| 0 | $e^x$ | 1 | 1 |
| 1 | $e^x$ | 1 | 1 |
| 2 | $e^x$ | 1 | $1/2$ |
| 3 | $e^x$ | 1 | $1/6$ |
| 4 | $e^x$ | 1 | $1/24$ |

COMPARE the graph of $e^x$ and the Taylor approximation to increasing values of $n$ on the calculator & discuss "interval of convergence"

Example $\quad f(x) = \ln(1 + \frac{1}{2}x)$; $\; p_4(x) = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 - \frac{1}{64}x^4$

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ | $\frac{f^{(k)}(0)}{k!}$ |
|---|---|---|---|
| 0 | $\ln(1+\frac{x}{2})$ | $\ln(1) = 0$ | 0 |
| 1 | $\frac{1}{2}(1+\frac{x}{2})^{-1}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | $-\frac{1}{4}(1+\frac{x}{2})^{-2}$ | $-\frac{1}{4}$ | $-\frac{1}{8}$ |
| 3 | $\frac{1}{4}(1+\frac{x}{2})^{-3}$ | $\frac{1}{4}$ | $\frac{1}{24}$ |
| 4 | $-\frac{3}{8}(1+\frac{x}{2})^{-4}$ | $-3/8$ | $-\frac{3}{8}\cdot\frac{1}{24} = -\frac{1}{64}$ |

$f(x)=\sqrt[4]{x+16}$; $p_2(x) = 2 + \frac{1}{32}x - \frac{3}{4096}x^2$

| K | $f^{(k)}(x)$ | $f^{(k)}(0)$ | $\frac{f^{(k)}(0)}{k!}$ |
|---|---|---|---|
| 0 | $(x+16)^{1/4}$ | $2$ | $2$ |
| 1 | $\frac{1}{4}(x+16)^{-3/4}$ | $\frac{1}{32}$ | $\frac{1}{32}$ |
| 2 | $-\frac{3}{16}(x+16)^{-7/4}$ | $-\frac{3}{16}\cdot 2^{-7}$ | $\frac{-3}{2048}\cdot\frac{1}{2}$ |

**Taylor Polynomial at *a***

The $n$th degree Taylor polynomial for $f$ at $x = a$ is

$$p_n(x)= f(a)+ f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

provided $f$ has $n$ derivatives at $a$.

$f(x)=e^{2x}$; $p_3(x)$ at 1 $= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3$

| K | $f^{(k)}(x)$ | $f^{(k)}(a)$ | $\frac{f^{(k)}(a)}{k!}$ |
|---|---|---|---|
| 0 | $e^{2x}$ | $e^2$ | $e^2$ |
| 1 | $2e^{2x}$ | $2e^2$ | $2e^2$ |
| 2 | $4e^{2x}$ | $4e^2$ | $2e^2$ |
| 3 | $8e^{2x}$ | $8e^2$ | $\frac{4}{3}e^2$ |
| $\cancel{4}$ | | | |

$f(x)=\ln(2-x)$; $p_4(x)$ at 1 $= -1(x-1) - \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

| K | $f^{(k)}(x)$ | $f^{(k)}(a)$ | $\frac{f^{(k)}(a)}{k!}$ |
|---|---|---|---|
| 0 | $\ln(2-x)$ | $0$ | $0$ |
| 1 | $-(2-x)^{-1}$ | $-1$ | $-1$ |
| 2 | $-(2-x)^{-2}$ | $-1$ | $-\frac{1}{2}$ |
| 3 | $-2(2-x)^{-3}$ | $-2$ | $-\frac{1}{3}$ |
| 4 | $-6(2-x)^{-4}$ | $-6$ | $-\frac{1}{4}$ |