Math 4355

Spring 2009 - Assignment 1

1. Show $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ is an IP for $\mathbb{C}^n$.

i) $\langle z, z \rangle = \sum_{j=1}^{n} \underbrace{|z_j|^2}_{\geq 0} \geq 0$

and equality holds if and only if $z_j = 0$ for all $j \in \{1, 2, \dots n\}$.

ii) $\overline{\langle z, w \rangle} = \overline{\sum_{j=1}^{n} z_j \overline{w_j}}$

$= \sum_{j=1}^{n} \overline{z_j \overline{w_j}} = \sum_{j=1}^{n} \overline{z_j} w_j$

$= \langle w, z \rangle$.

iii) $\langle cz, w \rangle = \sum_{j=1}^{n} c z_j \overline{w_j}$

$= c \cdot \sum_{j=1}^{n} z_j \overline{w_j} = c \langle z, w \rangle$

iv) $\langle z + v, w \rangle = \sum_{j=1}^{n} (z_j + v_j) \overline{w_j}$

$= \sum_{j=1}^{n} z_j \overline{w_j} + \sum_{j=1}^{n} v_j \overline{w_j}$

$= \langle z, w \rangle + \langle v, w \rangle$.

3. Let

$$\langle V, W \rangle = (\bar{w}_1, \bar{w}_2) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

Then for $V = (v_1, v_2)$ with $v_1 + 2v_2 = 0$,

$$\langle V, W \rangle = (\bar{w}_1, \bar{w}_2) \underbrace{\begin{pmatrix} v_1 + 2v_2 \\ 2v_1 + 4v_2 \end{pmatrix}}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$= (\bar{w}_1, \bar{w}_2) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 ,$$

in particular if $W = V$.

So, this is <u>not</u> an inner product, because

$\langle V, V \rangle = 0$ if, e.g. $V = (2, -1) \neq (0,0)$.

6. Given a sequence $\{f_n\}_{n=1}^{\infty}$

$$f_n(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{n} \\ 0, & \text{else} \end{cases}$$

then

$$\| f_n - 0 \| = \Big(\underbrace{\int_0^1 |f_n(t) - 0|^2 dt}_{\int_0^{\frac{1}{n}} 1^2 dt} \Big)^{\frac{1}{2}} = \sqrt{\tfrac{1}{n}} \xrightarrow{n \to \infty} 0$$

so $f_n \to 0$ in $L^2([0,1])$.

But

$$\max_{t\in[0,1]} |f_n(t) - 0| = 1 \quad , \quad \text{independent of } n$$

because $f_n(0) = 1$ and $f_n(t) \leq 1$ for all $t \in [0,1]$.

So $f_n \not\to 0$ uniformly.

7. Given a sequence $\{f_n\}_{n=1}^{\infty}$

$$f_n(t) = \begin{cases} \sqrt{n}, & 0 \leq t \leq \frac{1}{n^2} \\ 0, & \text{else} \end{cases}$$

then

$$\|f_n - 0\| = \Big(\underbrace{\int_0^1 |f_n(t) - 0|^2 dt}_{\int_0^{\frac{1}{n^2}} (\sqrt{n})^2 dt} \Big)^{\frac{1}{2}} = \sqrt{\frac{n}{n^2}} = \frac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0$$

so $f_n \to 0$ in $L^2([0,1])$.

But $f_n(0) = \sqrt{n} \to \infty$ so $\{f_n\}$ does not even converge pointwise!

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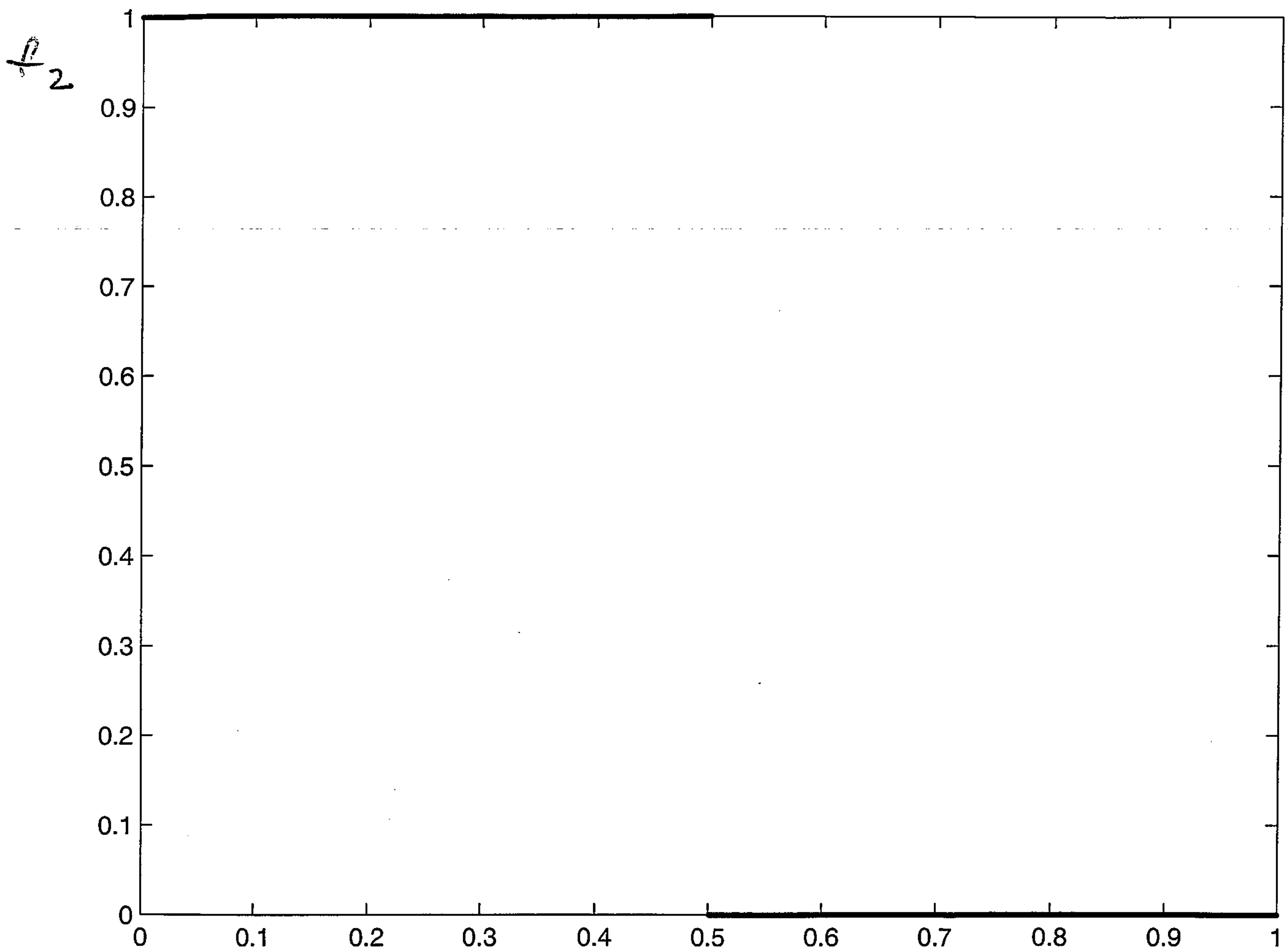
% Q6 and 7

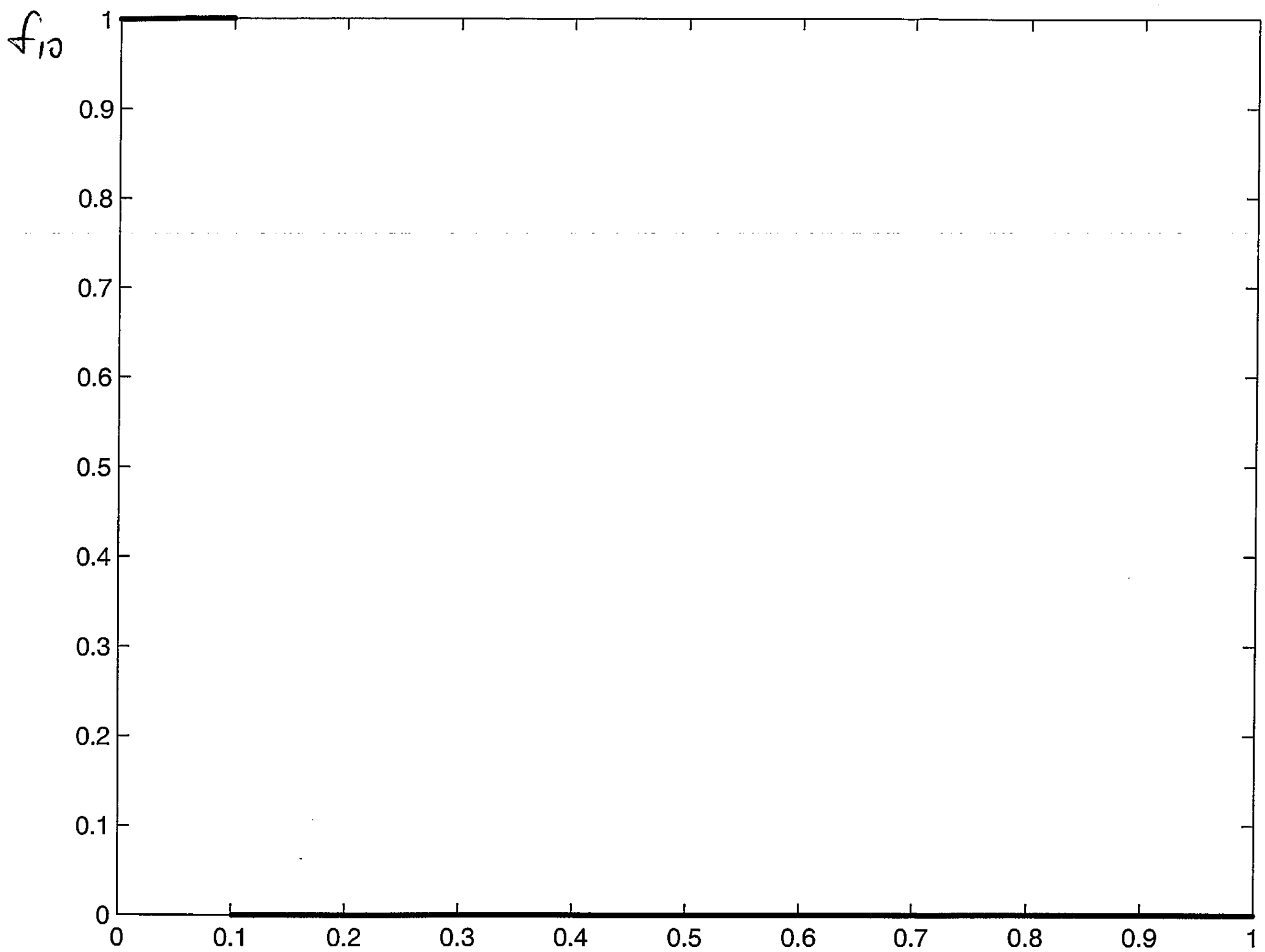
x=0:0.001:1;
f2=(x<=0.5*ones(1,1001));
figure, plot(x,f2,'.');
f10=(x<=0.1*ones(1,1001));
figure, plot(x,f10,'.');
f50=(x<=0.02*ones(1,1001));
figure, plot(x,f50,'.');

f2=sqrt(2)*(x<=(0.5)^2*ones(1,1001));
figure, plot(x,f2,'.');
f10=sqrt(10)*(x<=(0.1)^2*ones(1,1001));
figure, plot(x,f10,'.');
f50=sqrt(50)*(x<=(0.02)^2*ones(1,1001));
figure, plot(x,f50,'.');
```

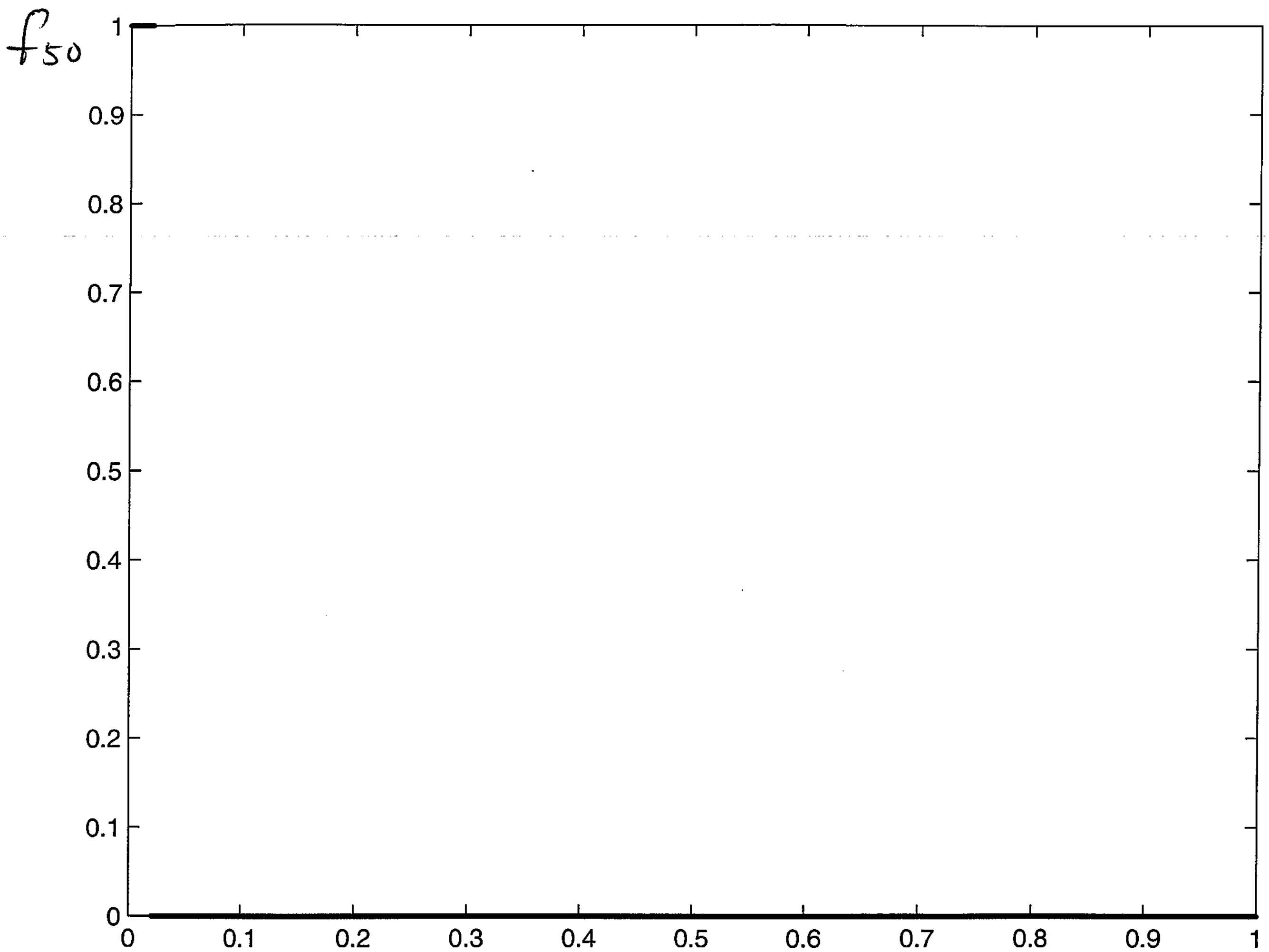
Question 6



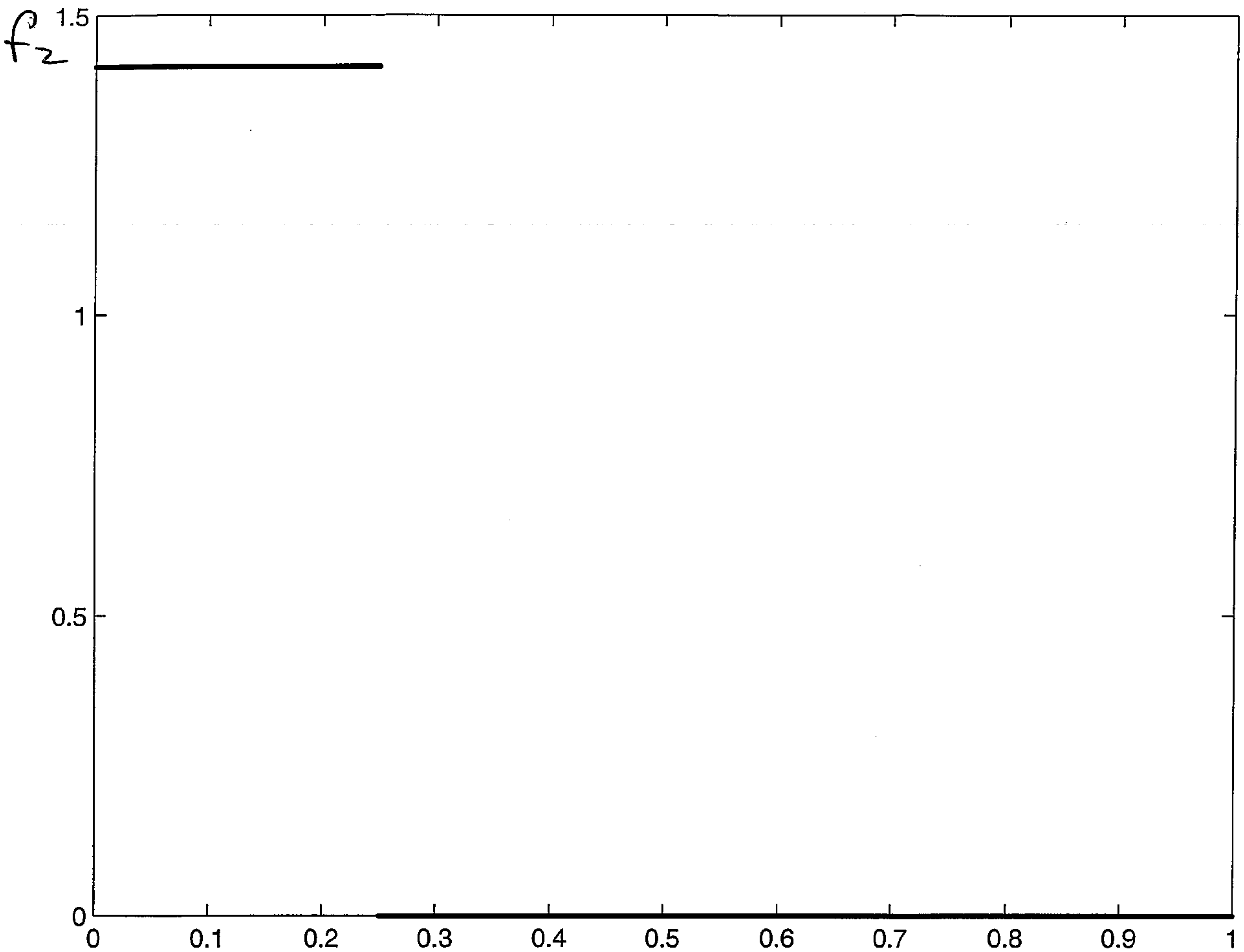
Q 6 (cont.)



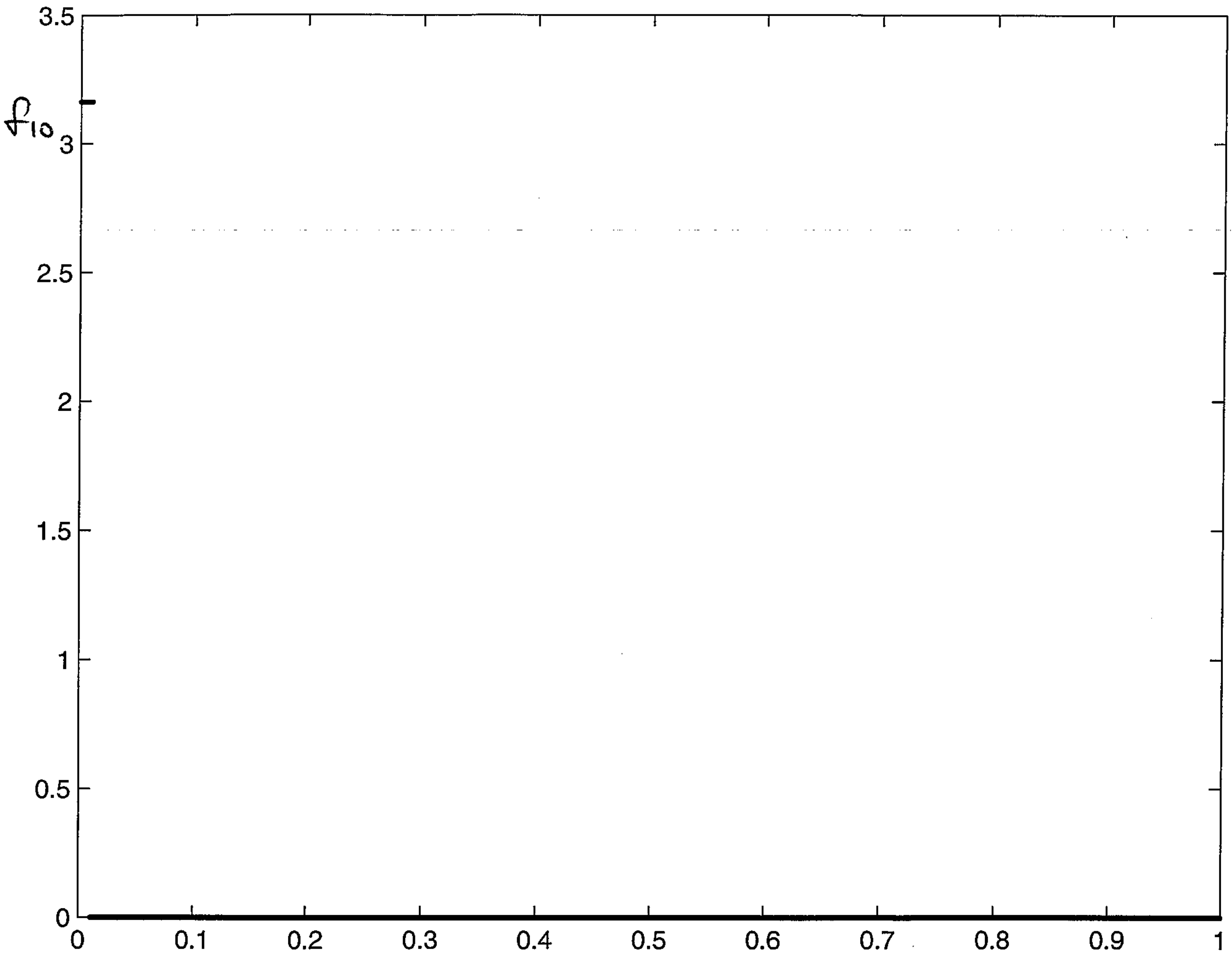
Q 6 (cont.)



Question 7



Q7 (cont.)



Q7 (cont.)