Mathematics of Signal Representations

Math4355 - Spring 2013 - Homework

Assignment 2, due Thursday, January 31

- Review 0.4, 0.5.1, 0.5.2 and to prepare for Exercise 26, read Section 0.7.1 up Theorem 0.34.
- Do Exercises, p. 36-37: 14; 15; 17; 26 (c) (assuming 26 (a) and (b) are true, solve for m and b in terms of the data x_j and y_j);
- (Matlab project) Download the linked file fitdata.mat (right click and save), load it into Matlab and plot the function **y(x)** using the Matlab command **plot** and the loaded vectors **x** and **y** containing the values for the independent variable x and the function value y. Include the plot in your homework. Following the strategy outlined in Section 0.7.1, write a program (matlab script) that will compute the least squares fit line **f(x)=mx+b** to approximate the "noisy" function values **y(x)** in fitdata.mat. Attach a printout of your program to your homework. Combine the plots for the function **y(x)** and the fit **f(x)** in one figure (e.g. using the Matlab command **hold on** after performing the first **plot**). Record the values of **m** and **b** computed by your fit program in your homework.

Math 4355

Assignment 2

14. From the lecture, we know that for each n,

$$\left\{ \frac{1}{\sqrt{2\pi}}, \quad \frac{\sin(jx)}{\sqrt{\pi}}, \quad \frac{\cos(jx)}{\sqrt{\pi}}, \quad 1 \le j \le n \right\}$$

is an orthonormal basis for V_n.

When computing (Fourier) coeffs, we note

$$\underset{\text{even}}{\underset{\uparrow}{\langle x^2}}, \ \frac{1}{\sqrt{\pi}} \underset{\text{odd}}{\underset{\uparrow}{\sin(jx)}} \rangle = 0 .$$

So, only need to compute

$$\langle x^2, \frac{1}{\sqrt{2\pi}} \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \frac{\pi^3}{3} = \frac{\sqrt{2}}{3} \pi^{5/2}$$

and

$$\langle x^2, \tfrac{1}{\sqrt{\pi}}\cos(jx)\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} x^2\cos(jx)\,dx$$

$$= \frac{2}{\sqrt{\pi}}\left(-\frac{\partial^2}{\partial j^2}\right)\underbrace{\int_0^{\pi} \cos(jx)\,dx}_{\frac{\sin(j\pi)}{j}}$$

$$= -\frac{2}{\sqrt{\pi}}\frac{\partial}{\partial j}\left(\frac{\pi\cos(j\pi)}{j} - \frac{\sin(j\pi)}{j^2}\right)$$

$$= -\frac{2}{\sqrt{\pi}}\left(-\cancel{\frac{\pi^2\sin(j\pi)}{j}} - 2\frac{\pi\cos(j\pi)}{j^2} + \cancel{2\frac{\sin(j\pi)}{j^3}}\right)$$

$$= +\frac{4\sqrt{\pi}}{j^2}\underbrace{\cos(j\pi)}_{(-1)^j}$$

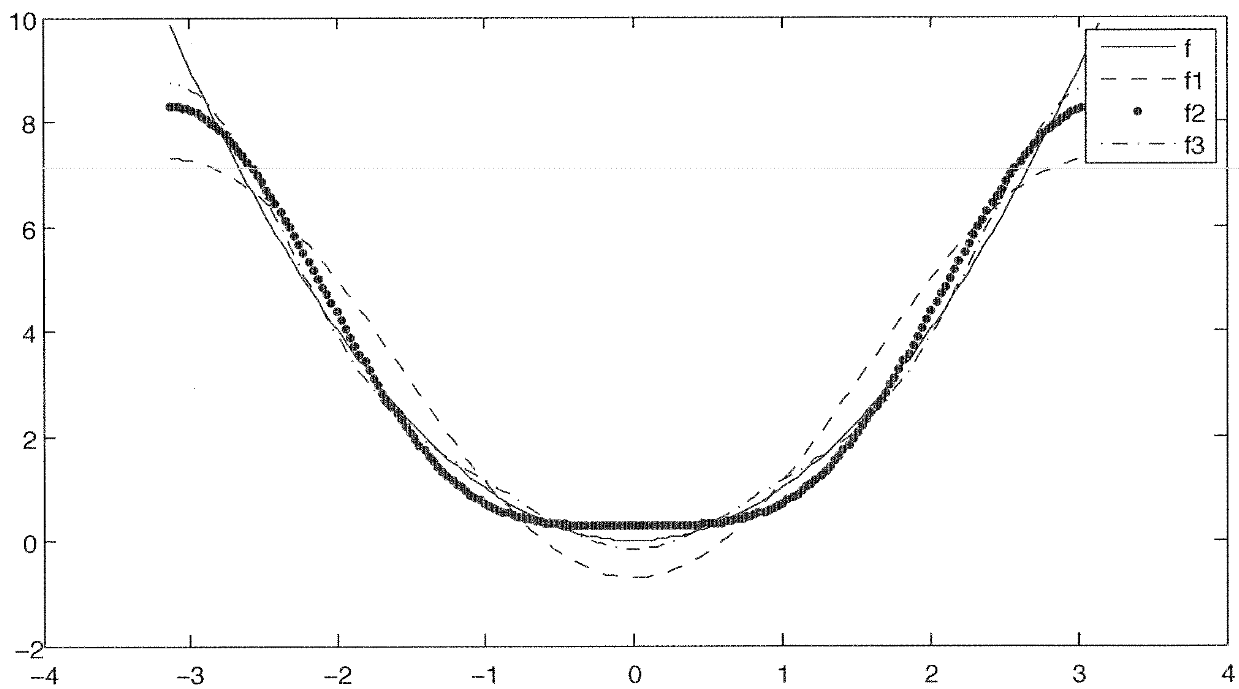
Consequently, if $\hat{f}_1, \hat{f}_2, \hat{f}_3$ are orthogonal projections of f onto V_1, V_2, V_3, then

$$\hat{f}_1(x) = \frac{\cancel{\sqrt{2}}}{3}\pi^{5/2}\frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} - 4\cancel{\sqrt{\pi}}\frac{1}{\cancel{\sqrt{\pi}}}\cos(x)$$

$$= \frac{\pi^2}{3} - 4\cos(x)$$

$$\hat{f}_2(x) = \hat{f}_1(x) + \cancel{\sqrt{\pi}}\frac{1}{\cancel{\sqrt{\pi}}}\cos(2x)$$

$$\hat{f}_3(x) = \hat{f}_2(x) - \frac{4}{9}\cos(3x)$$



10
8
6
4
2
0
−2
−4
−3
−2
−1
0
1
2
3
4
f
f1
f2
f3

Similarly, for $g(x) = x^3$, only "sin" terms contribute,

$$\langle x^3, \frac{1}{\sqrt{\pi}} \sin(jx) \rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x^3 \sin(jx)\, dx$$

$$= \frac{2}{\sqrt{\pi}} \frac{\partial^3}{\partial j^3} \underbrace{\int_0^{\pi} \cos(jx)\, dx}_{\frac{\sin(jx)}{j}}$$

$$= \ldots = \frac{2}{\sqrt{\pi}} \left(-\frac{\pi^3 \cos(j\pi)}{j} + \cancel{\frac{3\pi^2 \sin(j\pi)}{j^2}} + \frac{6\pi \cos(j\pi)}{j^3} - \cancel{\frac{6 \sin(j\pi)}{j^4}} \right)$$

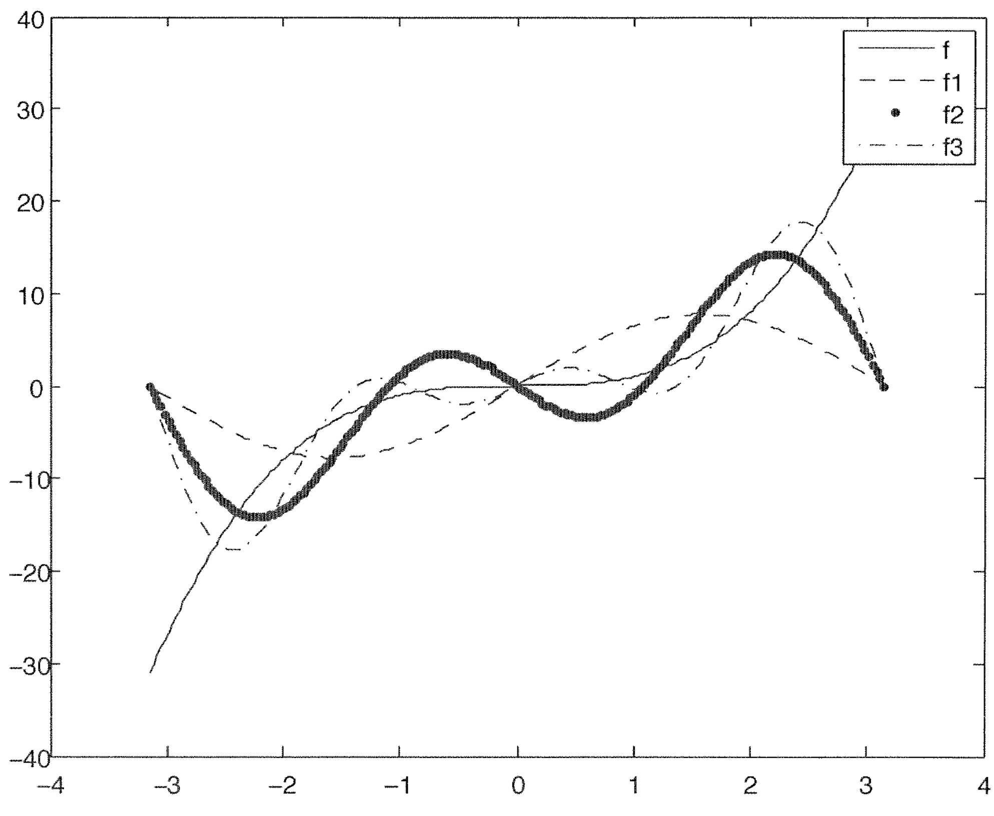
$$= -2\pi^{5/2} \frac{(-1)^j}{j} + \frac{12\pi^{1/2} (-1)^j}{j^3}$$

and the orthogonal projections $\hat{g}_1$, $\hat{g}_2$ and $\hat{g}_3$ onto V_1, V_2 and V_3 are

$$\hat{g}_1(x) = (2\pi^2 - 12) \sin(x)$$

$$\hat{g}_2(x) = \hat{g}_1(x) - \left(\pi^2 - \frac{3}{2} \right) \sin(2x)$$

$$\hat{g}_3(x) = \hat{g}_2(x) + \left(\frac{2}{3}\pi^2 - \frac{4}{9} \right) \sin(3x)$$



40
30
20
10
0
−10
−20
−30
−40
−4
−3
−2
−1
0
1
2
3
4
f
f1
f2
f3

15. We note that the functions $\{\phi(x), \psi(x), \psi(2x), \psi(2x-1)\}$ are orthogonal, b/c

$$\langle \phi, \psi \rangle = \int_0^{\frac{1}{2}} (1)\, dx + \int_{\frac{1}{2}}^{1} (-1)\, dx = 0$$

$$\langle \phi(x), \psi(2x) \rangle = \int_0^{\frac{1}{4}} (1)\, dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (-1)\, dx = 0$$

$$\langle \phi(x), \psi(2x-1) \rangle = \int_{\frac{1}{2}}^{3/4} (1)\, dx + \int_{3/4}^{1} (-1)\, dx = 0$$

$$\langle \psi, \psi(2x) \rangle = \int_0^{\frac{1}{4}} (1)\, dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (-1)\, dx = 0$$

$$\langle \psi, \psi(2x-1) \rangle = \int_{\frac{1}{2}}^{3/4} (-1)\, dx + \int_{3/4}^{1} (1)\, dx = 0.$$

$$\langle \psi(2x), \psi(2x-1) \rangle = 0$$ (nonzero on different intervals).

To get orthonormal basis, pick

$$\{\phi, \psi, \sqrt{2}\,\psi(2x), \sqrt{2}\,\psi(2x-1)\}.$$

Now, project using

$$\langle x, \phi \rangle = \int_0^1 x\, dx = \frac{1}{2}$$

$$\langle x, \psi \rangle = \int_0^{\frac{1}{2}} x\, dx - \int_{\frac{1}{2}}^{1} x\, dx = \frac{1}{4} - \frac{3}{8}$$

$$= -\frac{1}{8}$$

$$\langle x, \sqrt{2}\psi(2x) \rangle = \sqrt{2}\left(\int_0^{\frac{1}{4}} x\,dx - \int_{\frac{1}{4}}^{\frac{1}{2}} x\,dx \right)$$

$$= \sqrt{2}\left(\frac{1}{32} - \frac{3}{32} \right) = -\sqrt{2}\ \frac{1}{16}$$

$$\langle x, \sqrt{2}\,\psi(2x-1) \rangle = \sqrt{2}\ \left(\frac{5}{32} - \frac{7}{32} \right) = -\sqrt{2}\ \frac{1}{16}$$

which gives

$$\hat{f}(x) = \frac{1}{2} - \frac{1}{8}\psi(x) - \sqrt{2}\ \frac{1}{16}\ \sqrt{2}\,\psi(2x)$$

$$- \sqrt{2}\ \frac{1}{16}\ \sqrt{2}\ \psi(2x-1)$$

$$= \begin{cases} \frac{1}{8}, & 0 \le x < \frac{1}{4} \\ \frac{3}{8}, & \frac{1}{4} \le x < \frac{1}{2} \\ \frac{5}{8}, & \frac{1}{2} \le x < \frac{3}{4} \\ \frac{7}{8}, & \frac{3}{4} \le x < 1 \end{cases}.$$

17. If $\langle u_0, v \rangle = \langle u_1, v \rangle$ for all $v \in V$,

then

$$\langle u_0 - u_1, v \rangle = 0 \quad \text{for all } v \in V,$$

pick $v = u_0 - u_1$ gives

$$\langle u_0 - u_1, u_0 - u_1 \rangle = \|u_0 - u_1\|^2 = 0$$

$$\Rightarrow u_0 - u_1 = 0 \Rightarrow u_0 = u_1 .$$

2E.c) Assuming we know that

$$E(m,b) = \sum_{i=1}^{N} (m x_i + b - y_i)^2$$

is minimized if and only if

$$\frac{\partial E}{\partial m} = \frac{\partial E}{\partial b} = 0$$

then

$$\frac{1}{2}\frac{\partial E}{\partial m} = \cancel{2}\sum_{i=1}^{N} (m x_i + b - y_i)\, x_i = 0$$

and

$$\frac{1}{2}\frac{\partial E}{\partial b} = \cancel{2}\sum_{i=1}^{N} (m x_i + b - y_i) = 0 .$$

Solve for m, b

$$m \sum_{i=1}^{N} x_i^2 + b \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} x_i y_i \qquad \cancel{0} \qquad (1)$$

$$m \sum_{i=1}^{N} x_i + N b = \sum_{i=1}^{N} y_i \qquad \cancel{0} \qquad (2)$$

$$\Rightarrow \quad b = \frac{1}{N}\left(\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i\right) \qquad (2')$$

plug into 1^{st} eq.

$$m \sum_{i=1}^{N} x_i^2 + \frac{1}{N}\left(\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i\right) = \sum_{i=1}^{N} x_i y_i .$$

Write $\frac{1}{N}\sum_{i=1}^{N} y_i = \bar{y}$, $\frac{1}{N}\sum_{i=1}^{N} x_i = \bar{x}$,

$$\sum_{i=1}^{N} x_i y_i = \langle X, Y \rangle$$

then

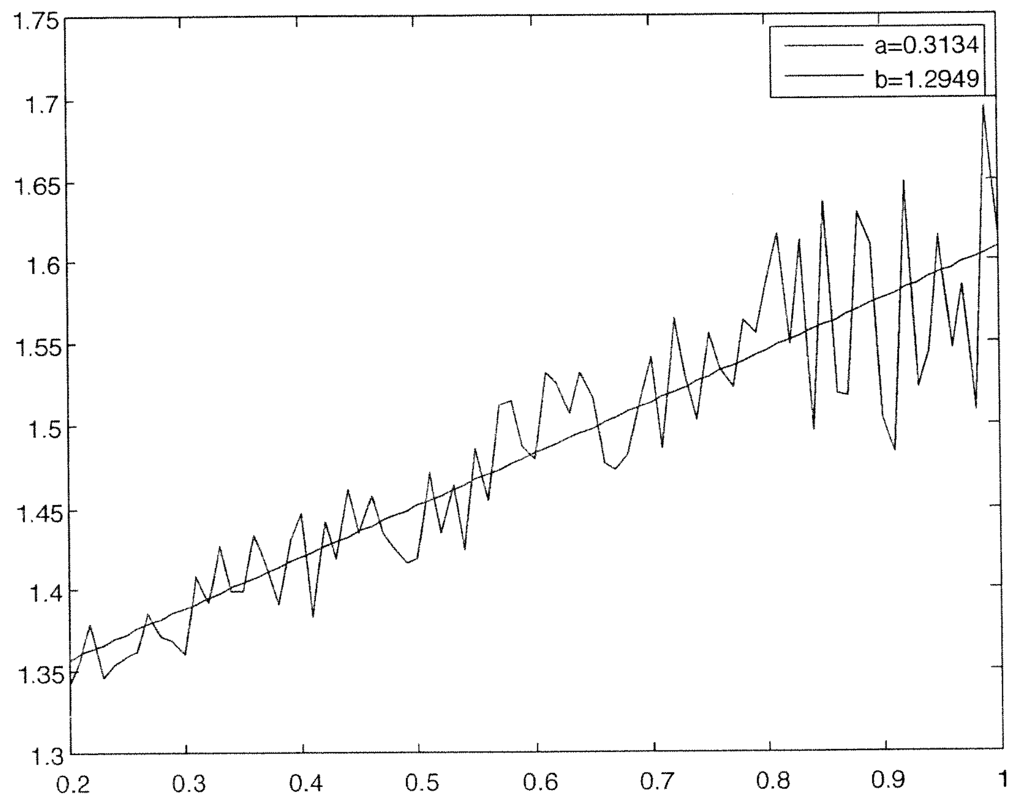
$$m\left(\sum_{i=1}^{N} x_i^2 - N\bar{x}^2\right) = \langle X, Y \rangle - N\bar{x}\bar{y}$$

$$\boxed{m = \frac{\langle X, Y \rangle - N\bar{x}\bar{y}}{\sum_{i=1}^{N} (x_i - \bar{x})^2}}$$

and in (2')

$$\boxed{b} = \bar{y} - \bar{x}\,\frac{\langle X, Y \rangle - N\bar{x}\bar{y}}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

$$= \frac{\bar{y}\left(\sum_{i=1}^{N} x_i^2 - N\bar{x}^2\right) - \bar{x}\left(\langle X, Y \rangle - \cancel{N\bar{x}\bar{y}}\right)}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$


a=0.3134
b=1.2949
1.75
1.7
1.65
1.6
1.55
1.5
1.45
1.4
1.35
1.3
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1