Mathematics of Signal Representations

Math4355 - Spring 2013 - Homework

Assignment 4, due Thursday, February 14

- Review Sections 1.2.1, 1.2.3 up to Lemma 1.7 and 1.2.4.

- Matlab project
  Write an m-file part_sum_fs.m in Matlab code which helps plot trigonometric polynomials of the form
  y(t) = a_0 + a_1*cos(t)+ b_1*sin(t) + a_2*cos(2*t)+ b_2*sin(2*t) + ... + a_n*cos(n*t)+ b_n*sin(n*t)

  The m-file (function) should be called as part_sum_fs(a,b,t) where the vectors containing the coefficients are
  a=[a_0, a_1 ,..., a_n]
  b=[b_1, b_2 ,..., b_n]
  and the row vector t contains times at which y(t) is evaluated.

  This should work for plotting y=1+2cos(t)+3sin(t)-sin(3t) on $[0,4\pi]$. To do this, we want to type
  ```
  >>a = [1,2,0,0];
  >>b = [3,0,-1];
  >>t = linspace(0,4*pi,500);
  >>y = part_sum_fs(a,b,t);
  >>plot(t,y)
  >>xlabel('t'), ylabel('y'), title('y=1+2cos(t)+3sin(t)-sin(3t)')
  ```

  Include the plot and your m file in your homework. Also include plots for the trigonometric polynomials
  y=sin(t)+sin(2t)/2+sin(3t)/3+sin(4t)/4
  y=cos(t)+cos(3t)/32+cos(5t)/52

- Do Exercises p. 84-85: 12; 13; 21; 22.

Math 4355

Assignment 4

12. Given $f(x) = e^{rx}, \quad -\pi \le x \le \pi$.

Fourier coeffs

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{rx}\,dx = \frac{1}{2\pi r}\left(e^{r\pi} - e^{-r\pi}\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{rx} \underbrace{\cos(nx)}_{\frac{1}{2}\left(e^{inx}+e^{-inx}\right)}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{r+in}\left(e^{(r+in)\pi} - e^{-(r+in)\pi}\right)\right.$$

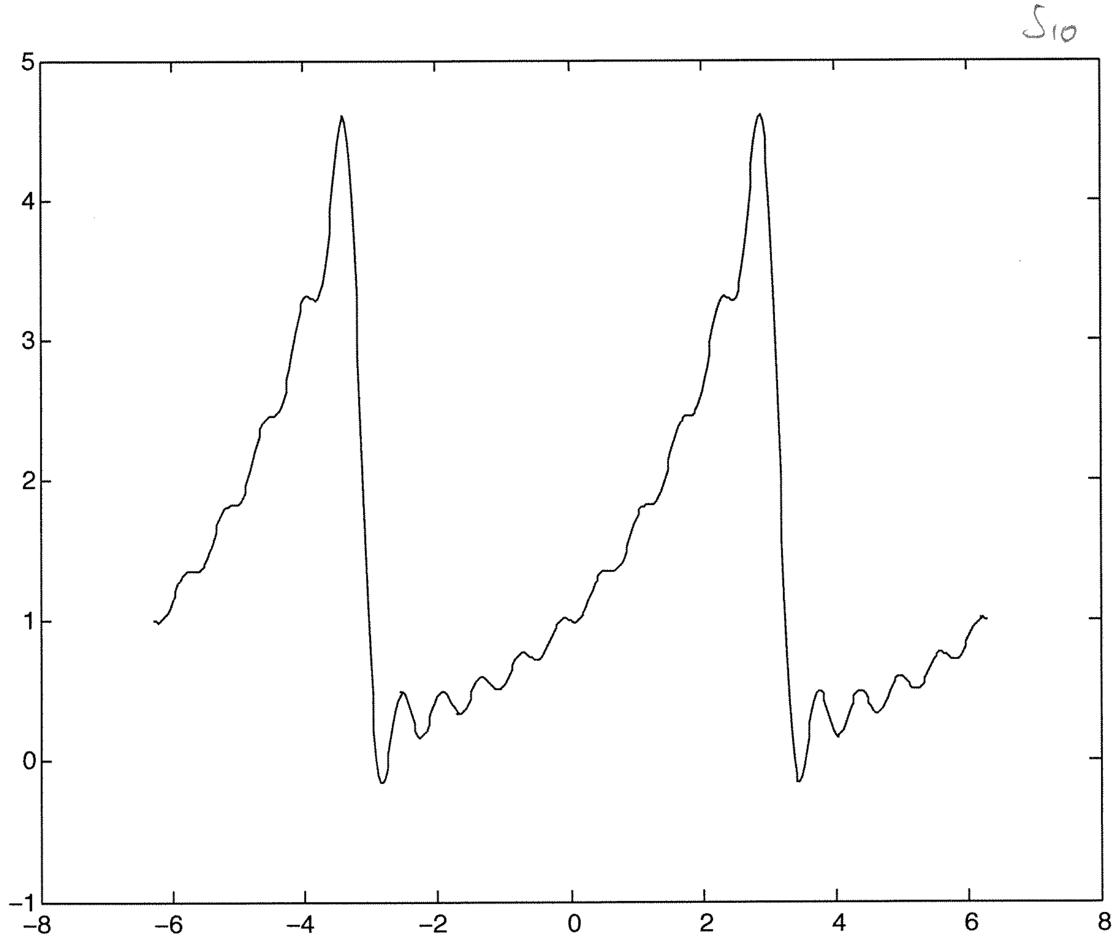
$$\left.+ \frac{1}{r-in}\left(e^{(r-in)\pi} - e^{-(r-in)\pi}\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{e^{r\pi} - e^{-r\pi}}{r+in}(-1)^n\right.$$

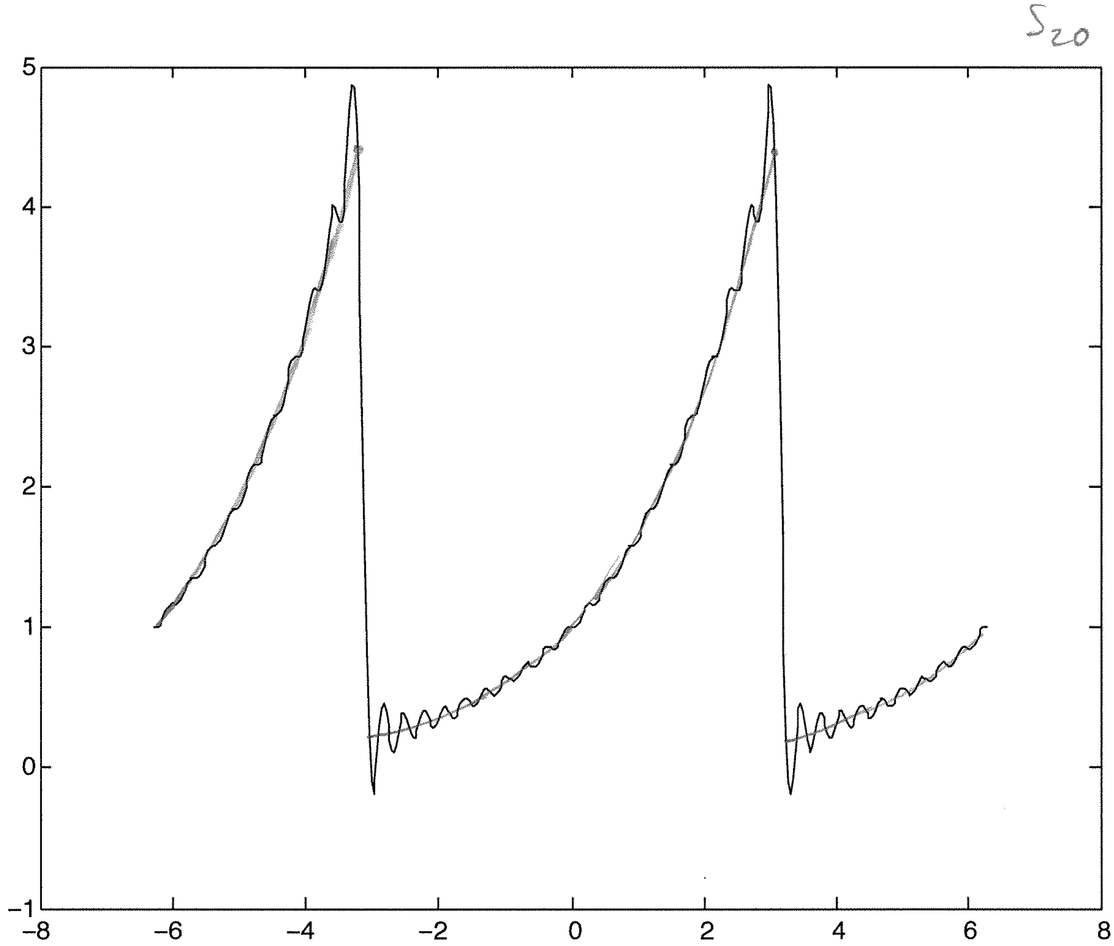
$$\left.+ \frac{e^{r\pi} - e^{-r\pi}}{r-in}(-1)^n\right)$$

$$= \frac{1}{\cancel{2}\pi}(-1)^n r\,\frac{e^{r\pi} - e^{-r\pi}}{r^2+n^2}.$$

Question 12
N=10

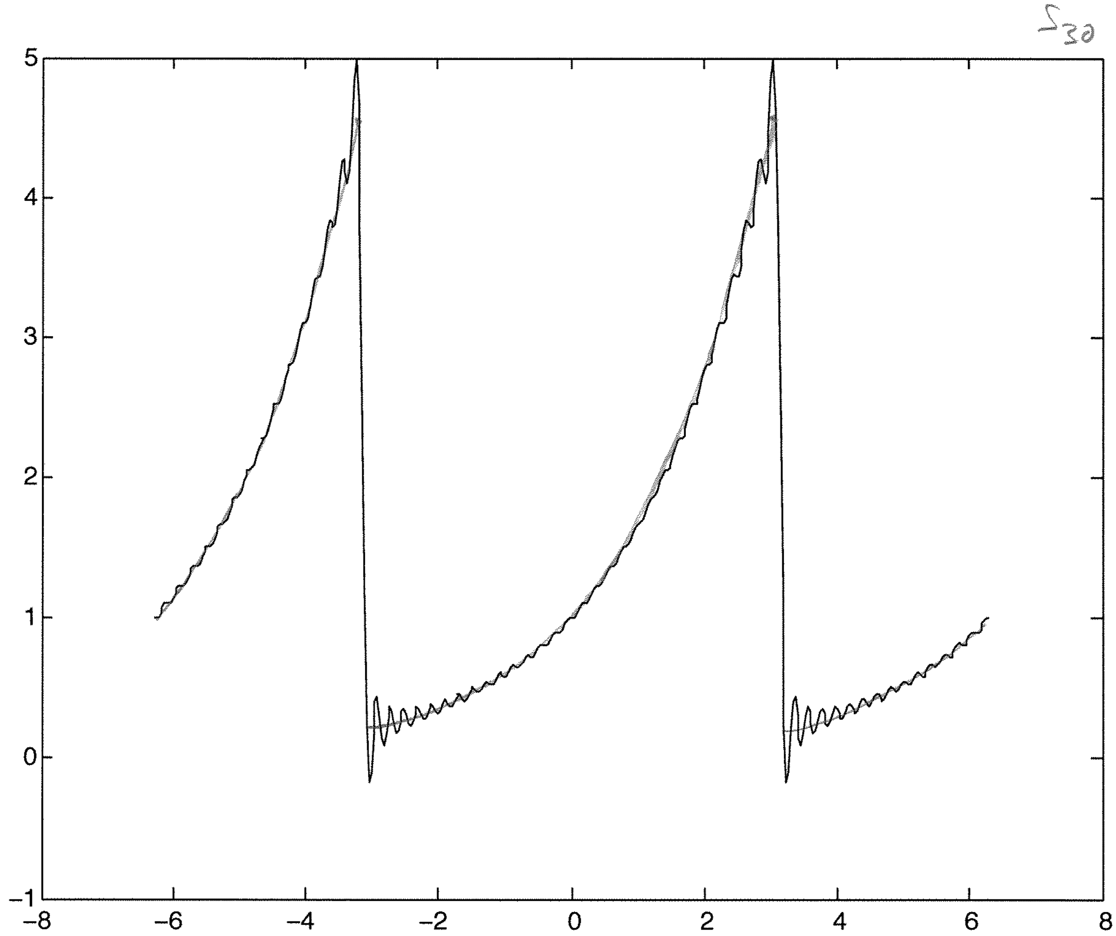


Question 12
N = 20



Question 12

$N = 30$



convergence is bad near discontinuity

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{rx} \underbrace{\sin(nx)}_{\frac{1}{2i}(e^{inx} - e^{-inx})} dx$$

$$= \frac{1}{\pi}(-1)^{n+1}\, n\left(e^{r\pi} - e^{-r\pi}\right)/(n^2+r^2).$$

Thus, the Fourier series is

$$F(x) = \frac{e^{r\pi} - e^{-r\pi}}{\pi}\left(\frac{1}{2r} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{r^2+n^2} r\cos(nx)\right.\right.$$

$$\left.\left. + \frac{(-1)^{n+1}}{r^2+n^2} n \sin(nx)\right)\right)$$

13. If $f(x) = \sinh(x)$, then because f is odd, all $a_k = 0$.

On the other hand,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(e^x - e^{-x}) \sin(nx)\, dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} e^x \sin(nx)dx \underbrace{- \int_{-\pi}^{\pi} e^{-x}\sin(nx)dx}_{+\int_{-\pi}^{\pi} e^{x'}\sin(nx')dx'}\right)$$

$$= \frac{1}{\pi} (-1)^{n+1} n \frac{e^{\cancel{r}\pi} - e^{-\pi}}{n^2 + 1},$$

same as in 12 with $r = 1$!

If $f(x) = \cosh(x)$, then all $b_k = 0$

and

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cancel{\tfrac{1}{2}}(e^x + \overset{\text{substitution}}{\cancel{e^{-x}}}) \, dx$$

$$= \frac{1}{2\pi} \left(e^{\pi} - e^{-\pi} \right)$$

$$a_k = \frac{1}{\pi} (-1)^k \frac{e^{\pi} - e^{-\pi}}{1 + n^2}.$$

21. a) Sawtooth fn is continuous and has left- and right derivatives for every x, so FS converges to $f(x)$ for every x, b/c

$$f(x) = \frac{1}{2}\left(f(x^-) + f(x^+)\right).$$

b) We have from Ex 1.10

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos((4k+2)x)$$

so choosing $x = \frac{\pi}{2}$ gives

$$\cos\left((4k+2)\frac{\pi}{2}\right) = \cos((2k+1)\pi) = -1$$

$\Rightarrow$

$$f(x) = x = \frac{\pi}{2}$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

22. By a theorem on unif. convergence, FS

for

$$f(x) = x^2 \quad \text{on} \quad [-\pi, \pi]$$

converges uniformly, because the

periodic extension of f is continuous

$\left(f(-\pi) = f(\pi) \right)$, and f' and f''

are continuous and bounded on $(-\pi, \pi)$.

Thus, we have for $x = \pi$

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n}$$

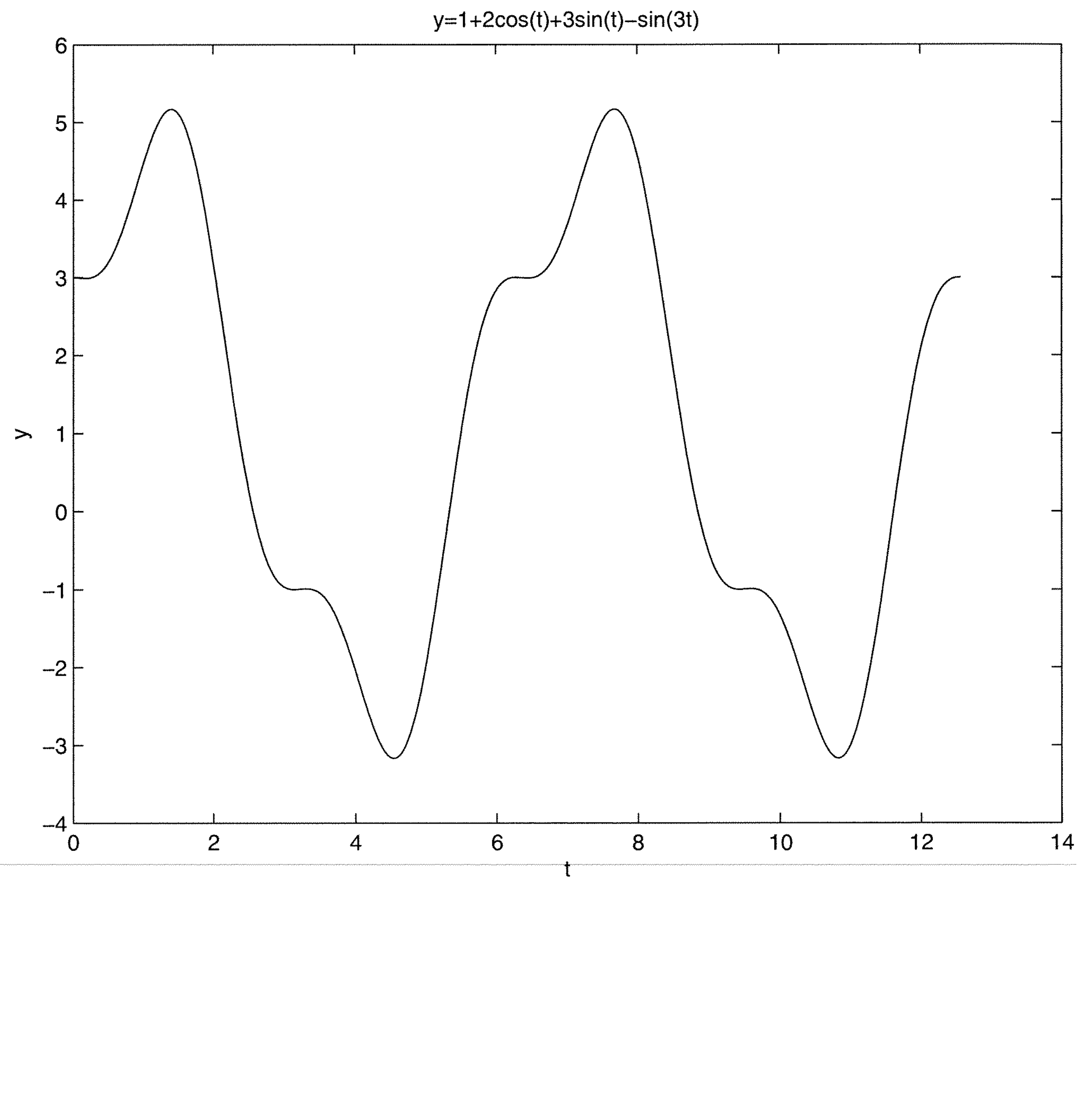
$$\frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

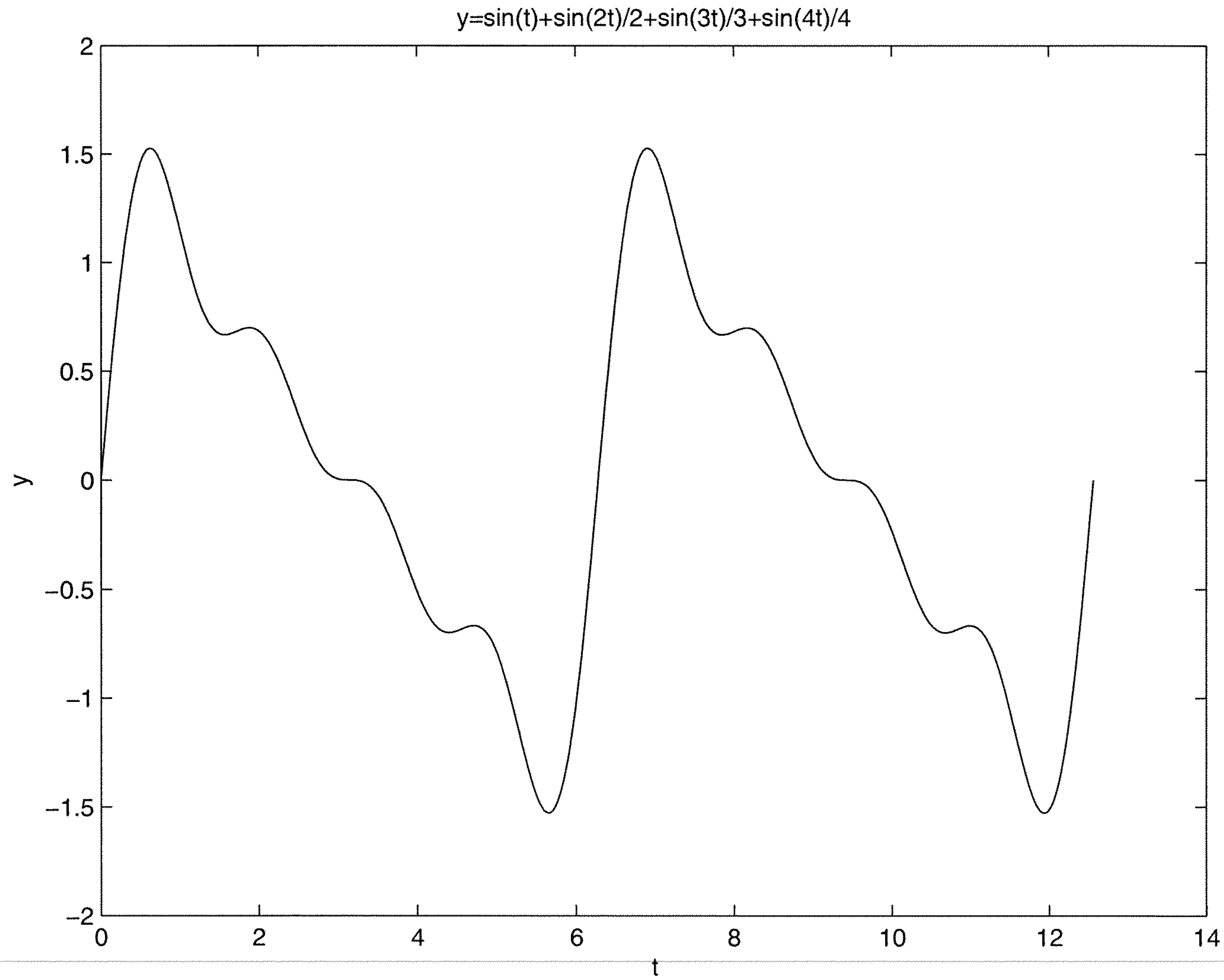
$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

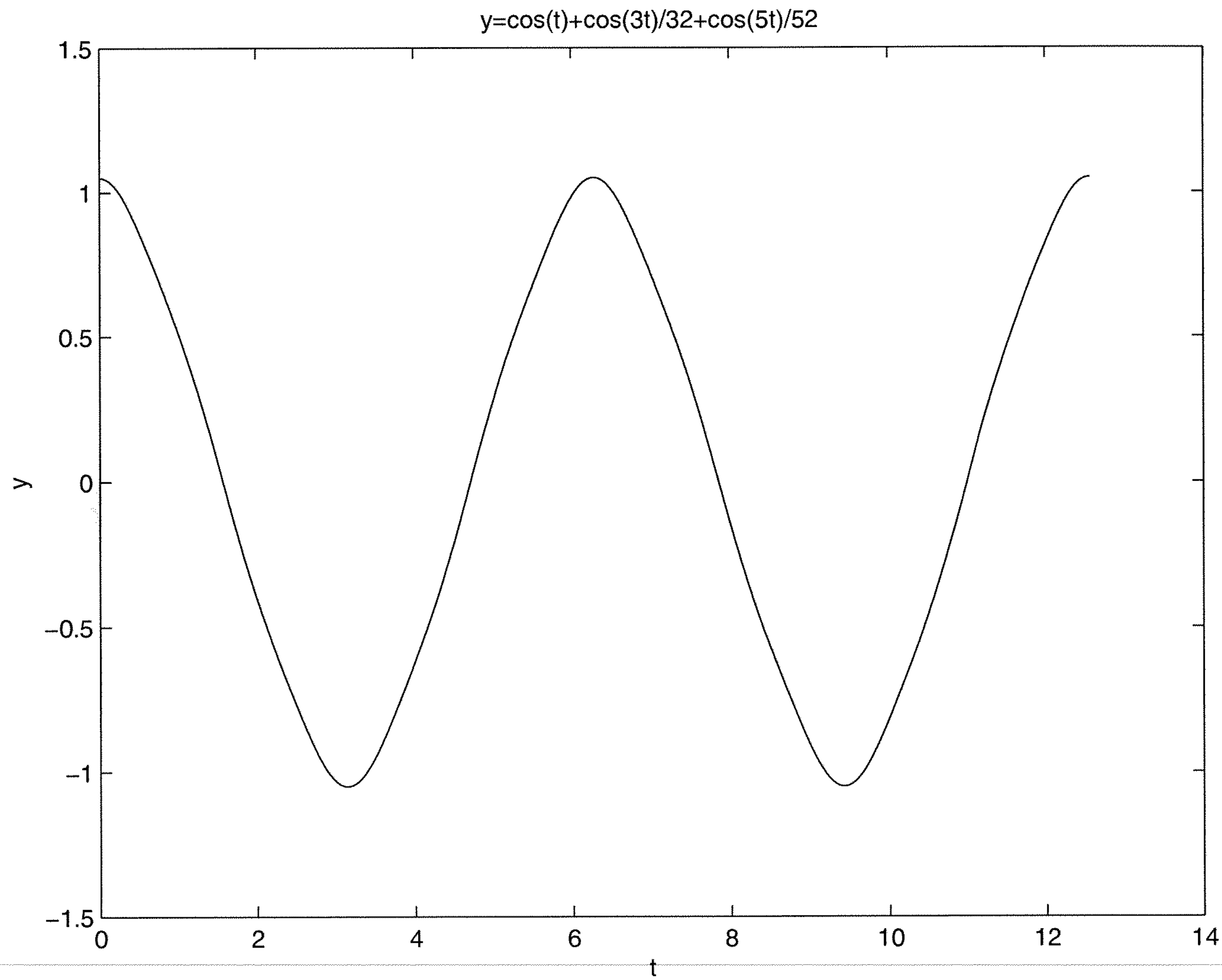
```
% function y=part_sum_fs(a,b,t)
%
% input: coefficient vectors a and b
%        points where FS is supposed to be
%        evaluated

function y=part_sum_fs(a,b,t)

n=length(a);
m=length(b);


% if vectors are not of compatible length,
% use only the piece where both coeffs exist
N=min([(n-1) m]);
T=length(t);

% constant part
y=a(1)*ones(1,T);

for k=1:1:N
    % index for a starts at two, for b at one
    y=y+a(k+1)*cos(k*t)+b(k)*sin(k*t);
end
```

Matlab project

Plots




y=sin(t)+sin(2t)/2+sin(3t)/3+sin(4t)/4
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
y
0
2
4
6
8
10
12
14
t


y=cos(t)+cos(3t)/32+cos(5t)/52
1.5
1
0.5
0
−0.5
−1
−1.5
y
0
2
4
6
8
10
12
14
t