

Math 4355

Assignment 4

1. Let $f(x) = x^2$ on $[-\pi, \pi]$.

Since f is even, $b_k = 0$ for all k .

Use same trick as in class to compute

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(kx) dx$$

$$- \frac{\partial^2}{\partial k^2} \cos(kx)$$

$$= -\frac{2}{\pi} \frac{\partial^2}{\partial k^2} \int_0^{\pi} \cos(kx) dx$$

$$\frac{\sin(k\pi)}{k}$$

$$= -\frac{2}{\pi} \frac{\partial}{\partial k} \left[\frac{\pi \cos(k\pi)}{k} - \frac{\sin(k\pi)}{k^2} \right]$$

$$= -\frac{2}{\pi} \left[-\frac{\pi^2 \sin(k\pi)}{k} - \frac{2\pi \cos(k\pi)}{k^2} + \frac{2 \sin(k\pi)}{k^3} \right]$$

$$= \frac{2}{\pi} \left[0 + \underbrace{\frac{2\pi \cos(k\pi)}{k^2}}_{2\pi \frac{(-1)^k}{k^2}} + 0 \right]$$

$$= (-1)^k \frac{4}{k^2} .$$

$$\text{So, } a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 = \frac{1}{2\pi} \left. \frac{x^3}{3} \right|_0^{\pi} = \frac{\pi^2}{3}$$

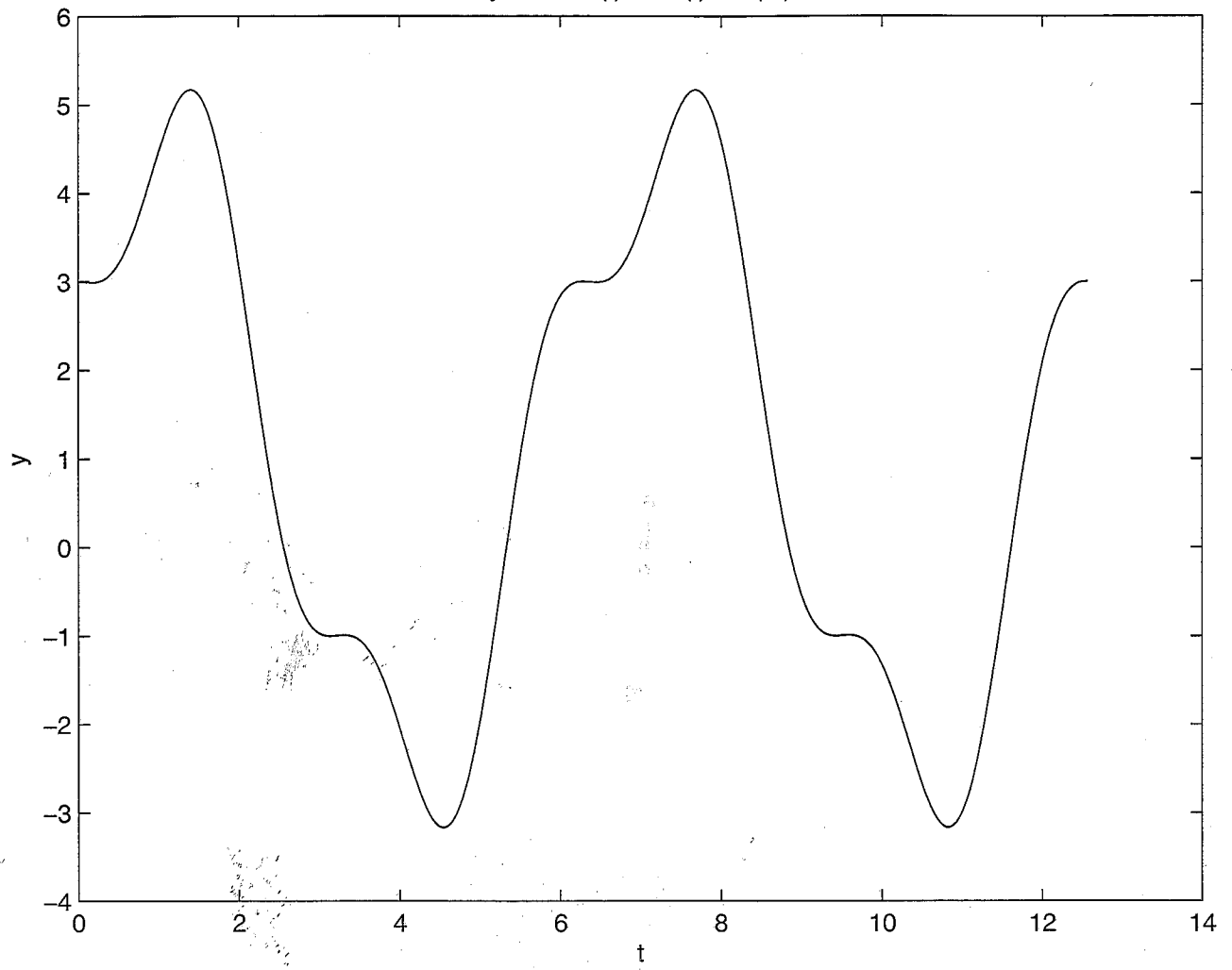
$$a_1 = -4, \quad a_2 = 1, \quad a_3 = -\frac{4}{9}$$

$$a_4 = \frac{1}{4}, \quad a_5 = -\frac{4}{25}, \quad a_6 = \frac{1}{9}, \quad a_7 = -\frac{4}{49}$$

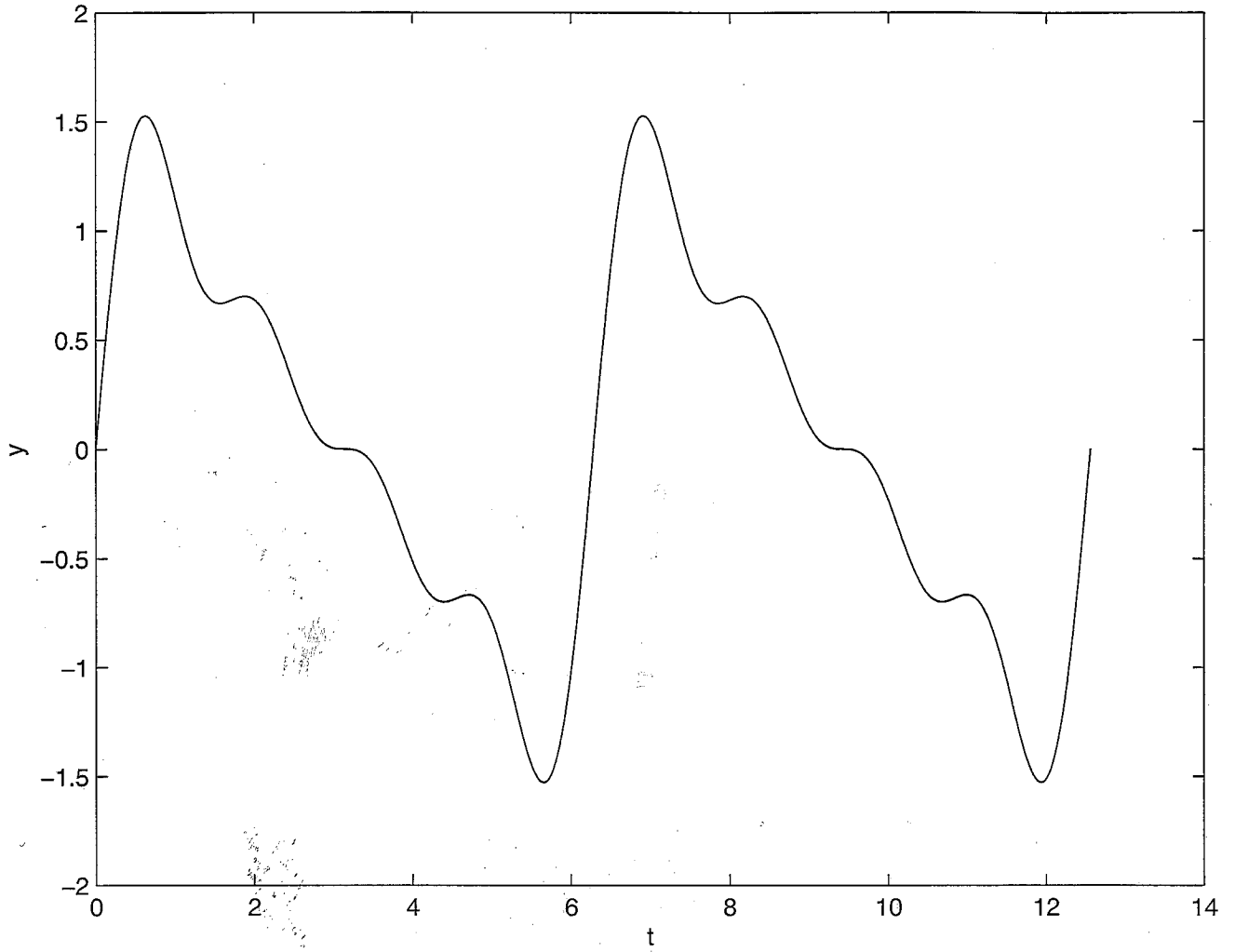
(Plots attached.)

7.

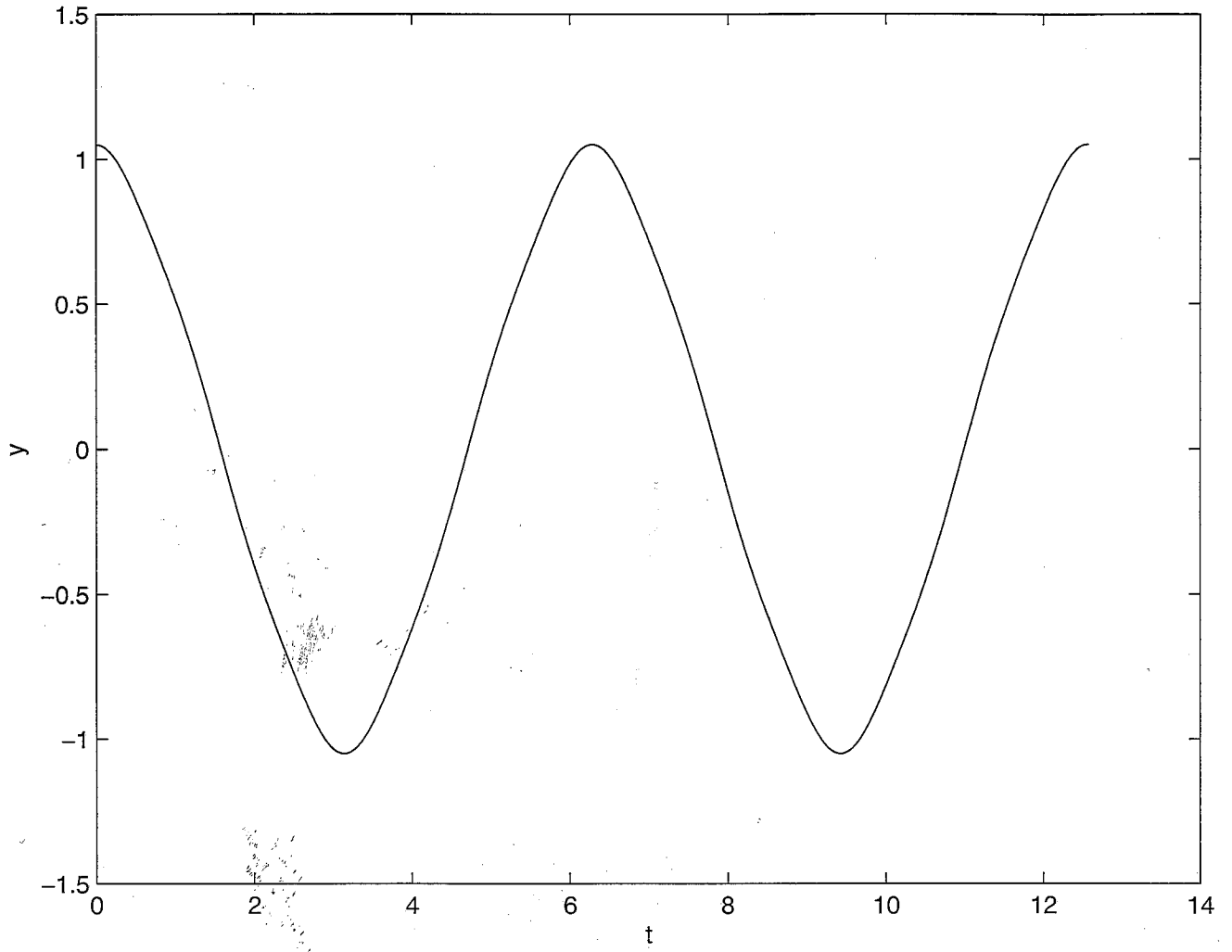
$$y=1+2\cos(t)+3\sin(t)-\sin(3t)$$

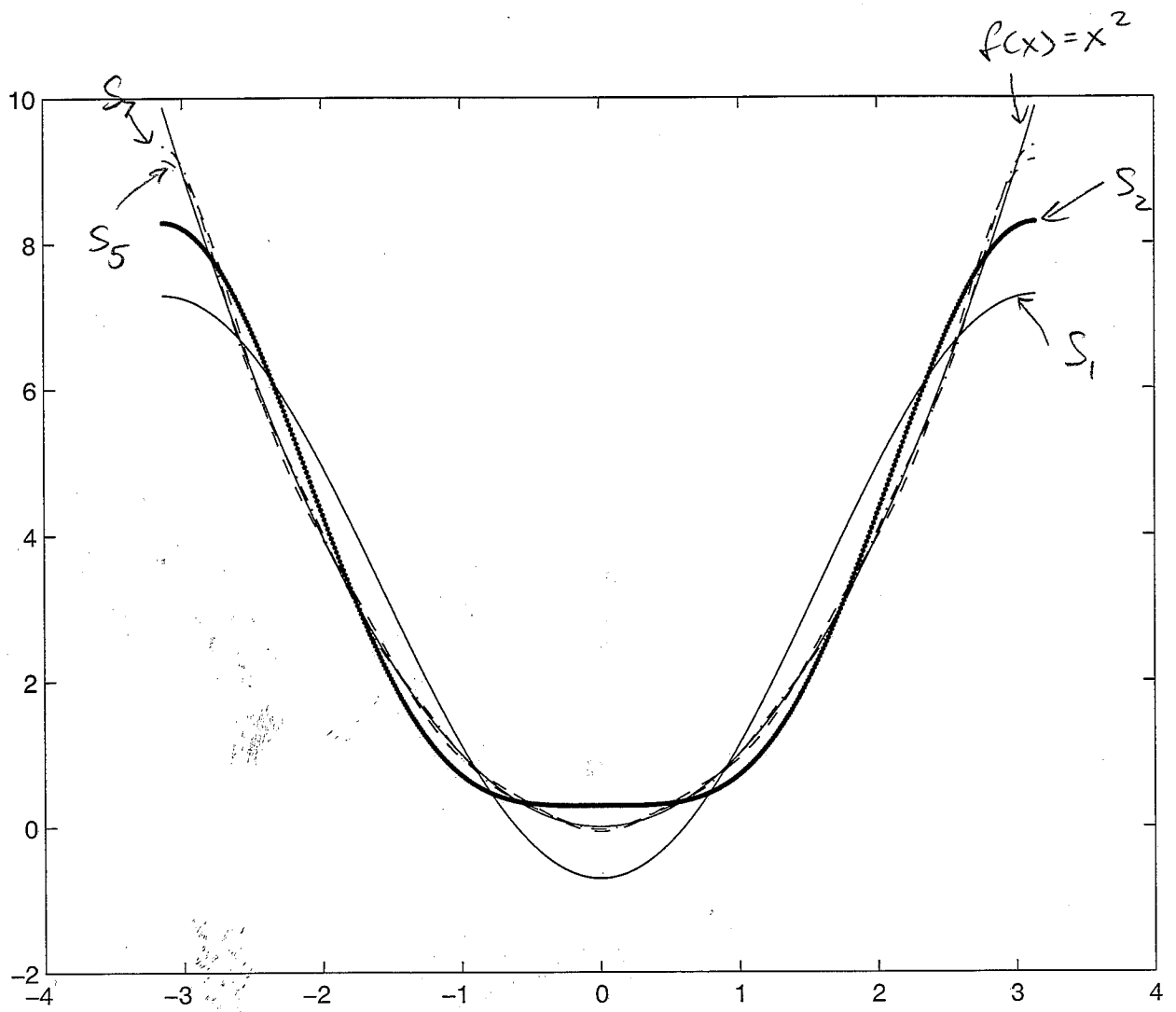


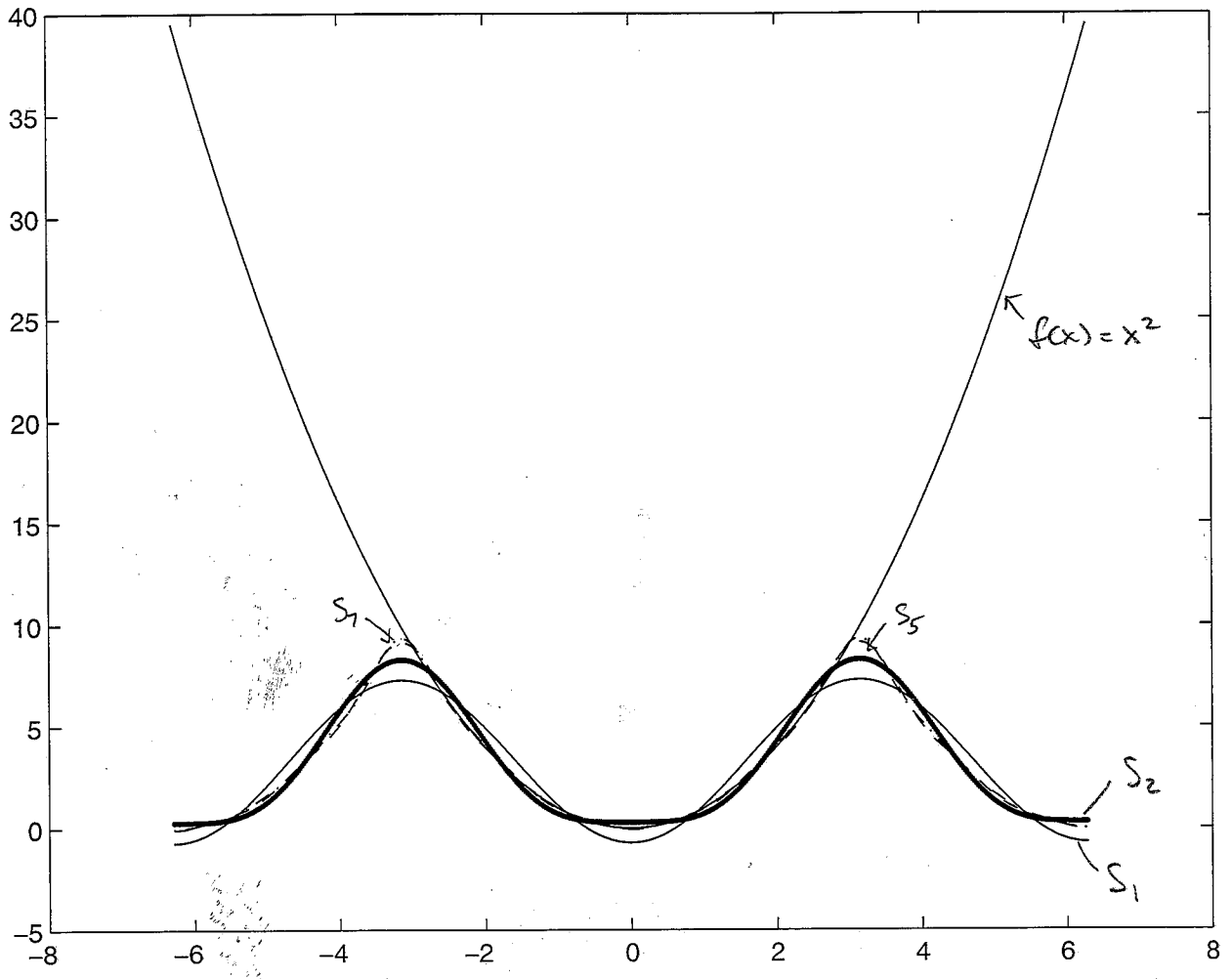
$$y = \sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \frac{\sin(4t)}{4}$$



$$y = \cos(t) + \cos(3t)/32 + \cos(5t)/52$$







7. FS for $f(x) = |\sin x|$ on $[-\pi, \pi]$

f even \Rightarrow all $b_k = 0$, $k \in \{1, 2, \dots\}$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| dx \stackrel{\text{Symm.}}{=} \frac{1}{\pi} \int_0^{\pi} \sin x dx$$

$$= \frac{1}{\pi} [-\cos x]_0^{\pi} = \frac{2}{\pi}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{\sin x \cos(kx)} dx$$

$$\frac{1}{2} (\sin((k+1)x) - \sin((k-1)x))$$

$$\frac{1}{\pi} \left[-\frac{\cos((k+1)x)}{k+1} + \frac{\cos((k-1)x)}{k-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{1 - \cos((k+1)\pi)}{k+1} - \frac{1 - \cos((k-1)\pi)}{k-1} \right]$$

$$\begin{aligned} \text{If } k \text{ odd, } k \pm 1 \text{ even} &\Rightarrow 1 - \cos((k \pm 1)\pi) = 0 \\ \text{even, } k \pm 1 \text{ odd} &\Rightarrow 1 - \cos((k \pm 1)\pi) = 2 \end{aligned}$$

So

$$\begin{aligned} a_{2k-1} &= 0 \quad \text{and} \quad a_{2k} = \frac{2}{\pi} \left[\frac{1}{2k+1} - \frac{1}{2k-1} \right] \\ &= -\frac{4}{\pi} \frac{1}{4k^2-1} \end{aligned}$$

and

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}$$

9. Given $f(x) = e^{rx}$, $-\pi \leq x \leq \pi$.

Fourier coeffs

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{rx} dx = \frac{1}{2\pi r} (e^{r\pi} - e^{-r\pi})$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{rx} \underbrace{\cos(nx)}_{\frac{1}{2}(e^{inx} + e^{-inx})} dx$$

$$= \frac{1}{2\pi} \left(\frac{1}{r+in} (e^{(r+in)\pi} - e^{-(r+in)\pi}) \right.$$

$$\left. + \frac{1}{r-in} (e^{(r-in)\pi} - e^{-(r-in)\pi}) \right)$$

$$= \frac{1}{2\pi} \left(\frac{e^{r\pi} - e^{-r\pi}}{r+in} (-1)^n + \frac{e^{r\pi} - e^{-r\pi}}{r-in} (-1)^n \right)$$

$$= \frac{1}{2\pi} (-1)^n r \frac{e^{r\pi} - e^{-r\pi}}{r^2 + n^2}$$

Similarly,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{rx} \underbrace{\sin(nx)}_{\frac{1}{2i}(e^{inx} - e^{-inx})} dx$$
$$= \frac{1}{\pi} (-1)^{n+1} n \frac{e^{r\pi} - e^{-r\pi}}{n^2 + r^2}$$

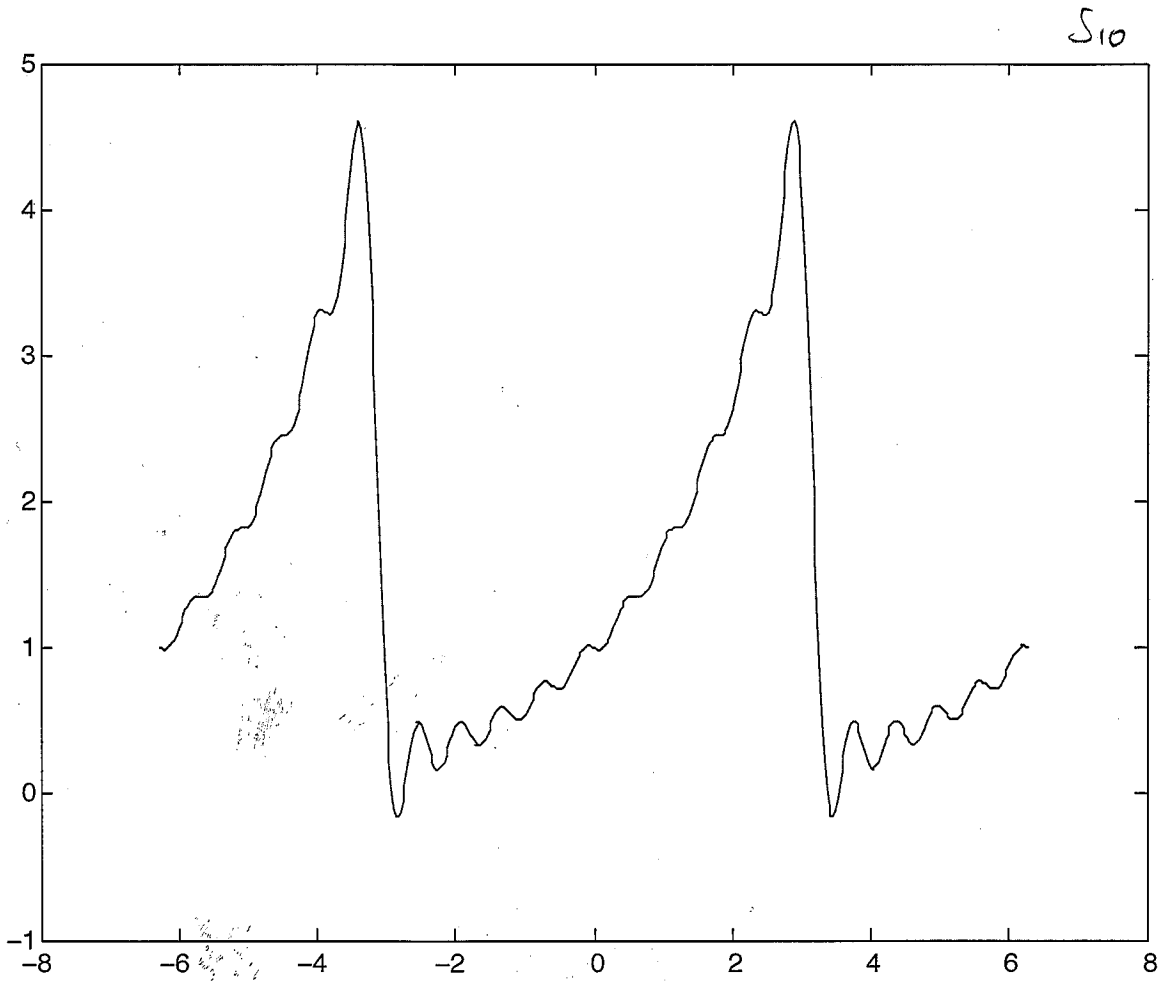
Fourier series

$$F(x) = \frac{e^{r\pi} - e^{-r\pi}}{\pi} \left(\frac{1}{2r} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n r \cos(nx)}{n^2 + r^2} + \frac{(-1)^{n+1} n \sin(nx)}{n^2 + r^2} \right) \right)$$

15. By convergence theorem,

$$F(x) = \begin{cases} 0 & \text{on } [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1] \\ \frac{1}{2} & \text{on } \left\{ -\frac{1}{2}, \frac{1}{2} \right\} \\ 1 & \text{on } \left(-\frac{1}{2}, \frac{1}{2}\right) \end{cases}$$

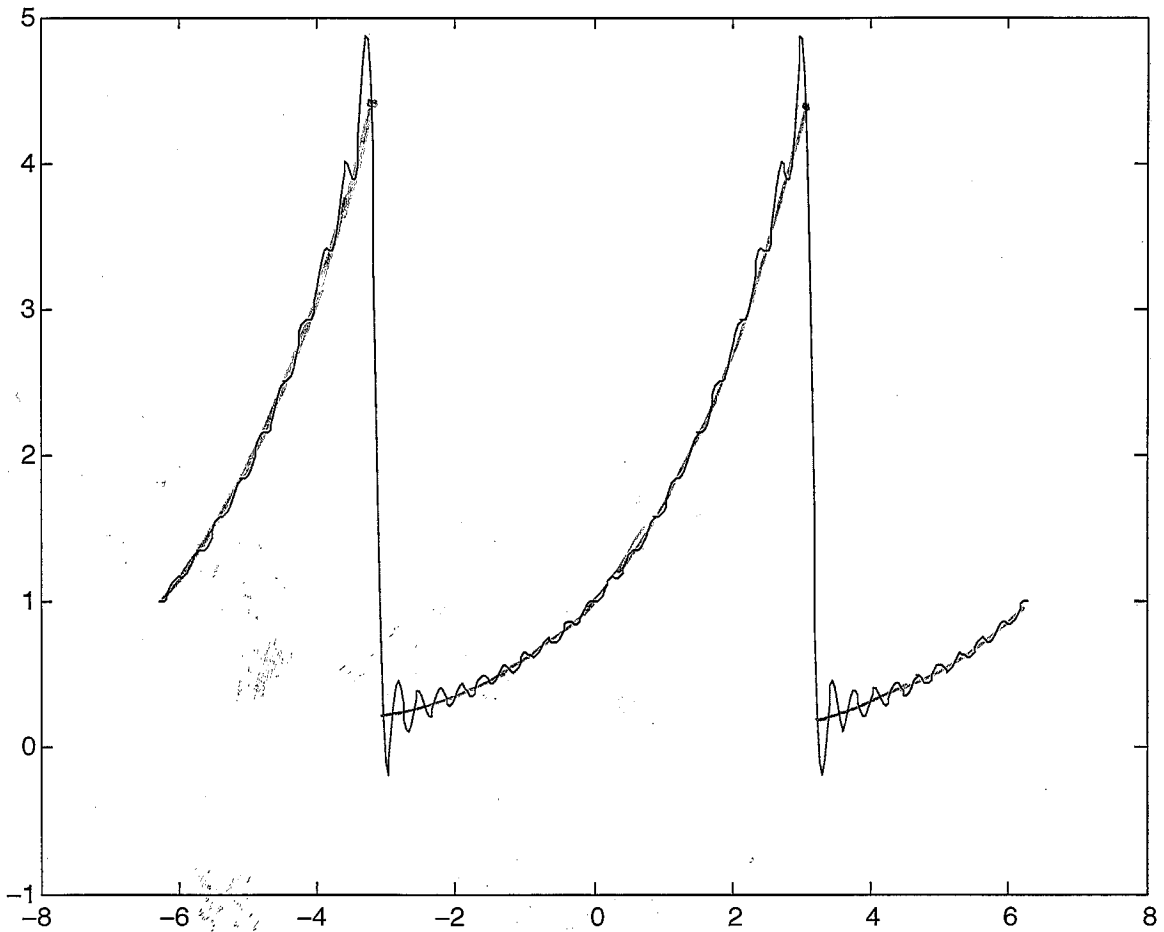
Question 9
 $N=10$



Question 9

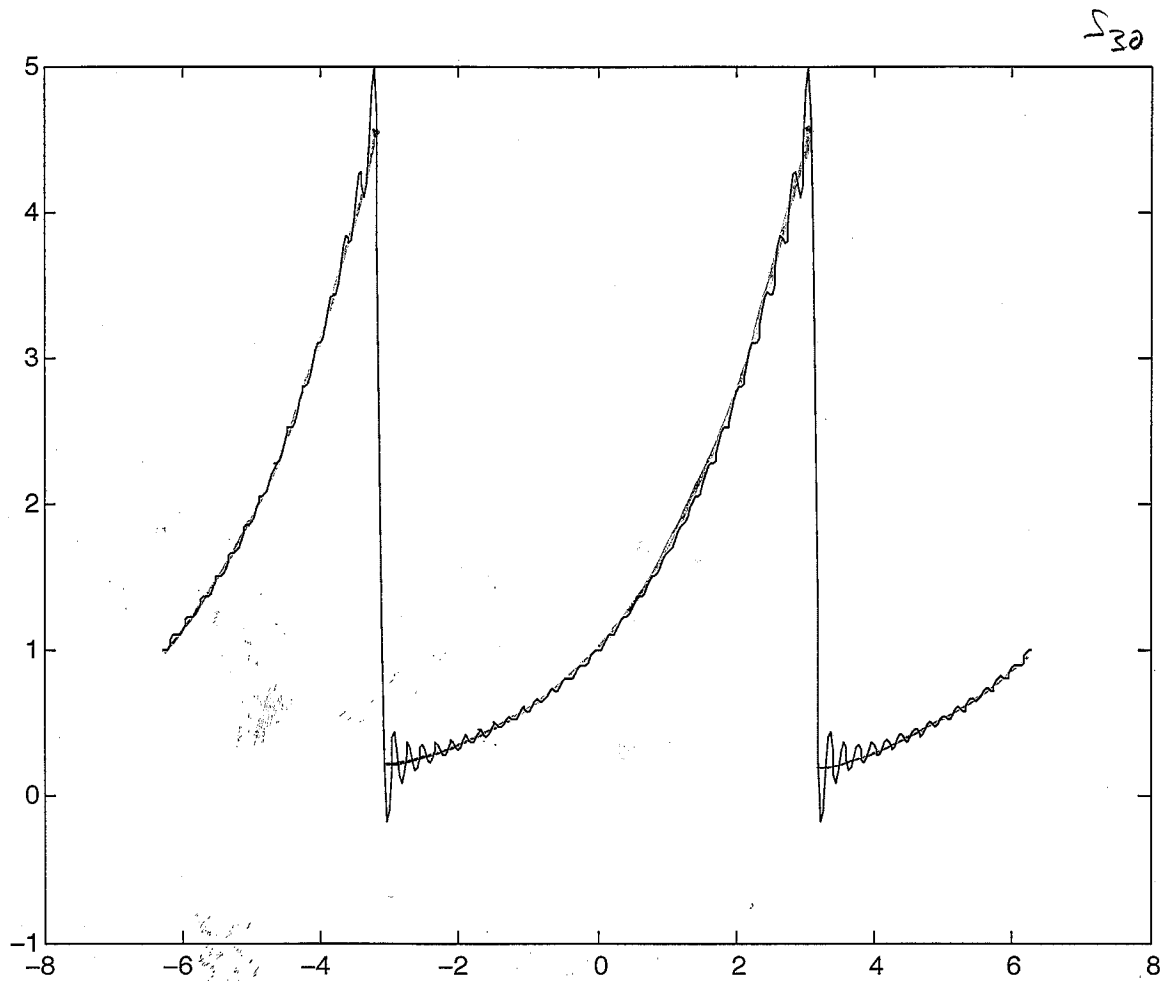
$N=20$

S_{20}



Question 9

$N=30$



convergence is bad near discontinuity

17. a) Sawtooth f_n is continuous and has left- and right derivatives for every x , so FS converges to $f(x)$ for every x , b/c

$$f(x) = \frac{1}{2} (f(x^-) + f(x^+))$$

b) We have from Ex 1.10

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos((4k+2)x)$$

so choosing $x = \frac{\pi}{2}$ gives

$$\cos((4k+2)\frac{\pi}{2}) = \cos((2k+1)\pi) = -1$$

\Rightarrow

$$\begin{aligned} f(x) = x &= \frac{\pi}{2} \\ &= \frac{\pi}{4} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \end{aligned}$$

\Rightarrow

$$\frac{\pi}{4} = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$