Math 4355

Assignment # 5

1.4 : 29)

Parseval (real form)

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = 2|a_0|^2 + \sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)$$

Here $f(x) = x^2$ gives FS with coeffs

$$a_0 = \frac{\pi^2}{3}, \quad a_n = (-1)^n\frac{4}{n^2}, \quad b_n = 0$$

so

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = 2\frac{\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\|$$

$$\frac{2}{\pi}\int_0^{\pi} x^4\,dx$$

$$\|$$

$$\frac{2}{\cancel{\pi}}\frac{\pi^{\cancel{5}4}}{5} \Rightarrow \frac{8\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}.$$

2.6 : 1) Let $f(t) = \begin{cases} \cos(3t), & -\pi \le t \le \pi \\ 0 & , \text{else} \end{cases}$

Compute for m integer

$$\int_{-\pi}^{\pi} \underbrace{\cos(mt)\cos(\lambda t)}_{\frac{1}{2}\left[\cos(mt+\lambda t) + \cos(mt-\lambda t)\right]}\, dt$$

$$\overset{\text{symm}}{=} \int_{0}^{\pi} \left(\cos((m+\lambda)t) + \cos((m-\lambda)t)\right) dt$$

$$= \left[\frac{\sin((m+\lambda)t)}{m+\lambda} + \frac{\sin((m-\lambda)t)}{m-\lambda} \right]_0^{\pi}$$

$$= \frac{\sin(m\pi + \lambda\pi)}{m+\lambda} + \frac{\sin(m\pi - \lambda\pi)}{m-\lambda}$$

$$= (-1)^m \left(\frac{\sin(\lambda\pi)}{m+\lambda} - \frac{\sin(\lambda\pi)}{m-\lambda} \right)$$

$$= (-1)^{m+1} \frac{2\lambda \sin(\lambda\pi)}{m^2 - \lambda^2} .$$

Now, Fourier transform of f,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos(3t) \cos(\omega t) \, dt$$

$-i \sin(\omega t)$ does not contribute!

$$= \frac{1}{\sqrt{2\pi}} \frac{2\lambda \sin(\pi\lambda)}{9-\lambda^2}$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\lambda \sin(\pi\lambda)}{\lambda^2-9} .$$

2) Similarly for $f(t) = \begin{cases} \sin(3t), & -\pi \le t \le \pi \\ 0, & \text{else} \end{cases}$

Compute

$$\int_{-\pi}^{\pi} \sin(mt) \sin(\lambda t) \, dt$$

$$= \int_{0}^{\pi} \left(\cos((m-\lambda)t) - \cos((m+\lambda)t) \right) dt$$

$$= \frac{\sin((m-\lambda)\pi)}{m-\lambda} - \frac{\sin((m+\lambda)\pi)}{m+\lambda}$$

$\sin(n\pi) = (-1)^n$

$$= (-1)^{m+1} \left(\frac{\sin(\lambda\pi)}{m-\lambda} + \frac{\sin(\lambda\pi)}{m+\lambda} \right)$$

$$= (-1)^{m+1} \frac{2m \sin(\lambda\pi)}{m^2-\lambda^2} .$$

So, FT

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin(3t)\,(-i\sin(\omega t))\,dt$$

$$= -\frac{i}{\sqrt{2\pi}} \frac{6\sin(\lambda\pi)}{9-\lambda^2}$$

$$= -i\sqrt{\frac{2}{\pi}} \frac{\sin(\lambda\pi)}{9-\lambda^2}.$$

FT rules:

Compute FT of $f(t) = e^{-|t|}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} e^{-t} e^{-i\omega t}\,dt + \int_{-\infty}^{0} e^{t} e^{-i\omega t}\,dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}.$$

i) FT of $f(t) = t^2 e^{-|t|}$

$$\mathcal{F}[t^2 f(t)] = -\frac{d^2}{d\omega^2} \mathcal{F}[f(t)](\omega)$$

$$= -\frac{d^2}{d\omega^2} \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

$$= -\sqrt{\frac{2}{\pi}} \frac{d}{d\omega} \left(-\frac{2\omega}{(1+\omega^2)^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2}{(1+\omega^2)^2} - \frac{4\omega^2}{(1+\omega^2)^3} \right)$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{1-\omega^2}{(1+\omega^2)^3}$$

ii) $f(t) = e^{-|t-2|}$

$$\mathcal{F}[f(t-a)](\omega) = e^{-i\omega a} \mathcal{F}[f(t)](\omega)$$

$$\rightarrow \hat{f}(\omega) = e^{-2i\omega} \mathcal{F}[e^{-|t|}](\omega)$$

$$= e^{-2i\omega} \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$