Math 4355
Assignment #6

5. We compute

$$(\phi * \phi)(x) = \int_{-\infty}^{\infty} \phi(y)\,\phi(x-y)\,dy$$

$$= \int_{0}^{1} (1)\,\phi(x-y)\,dy$$

distinguish cases

a) $x \leq 0 \Rightarrow x - y \leq -y \leq 0$

$\Rightarrow \phi(x-y) = 0$

$\Rightarrow (\phi * \phi)(x) = 0$

b) $x \geq 2 \Rightarrow x - y \geq 2 - y \geq 1$

$\Rightarrow \phi(x-y) = 0$

$\Rightarrow (\phi * \phi)(x) = 0$

c) $0 \leq x \leq 1$:

we need to show $(\phi * \phi)(x)$

$= 1 - |x-1| = 1 + x - 1$

$= x$

$$(\phi * \phi)(x) = \int_{0}^{x} (1)\,\phi(x-y)\,dy$$

$0 \leq x - y \leq x \leq 1$

$\Rightarrow \phi(x-y) = 1$

$$\Rightarrow \quad (\phi * \phi)(x) = \int_0^x 1 \, dy = x$$

d) $1 \leq x \leq 2$

We want $(\phi * \phi)(x) = 1 - |x-1|$

$$= 1 - (x-1)$$

$$= 2 - x$$

$$(\phi * \phi)(x) = \int_{x-1}^{1} \phi(x-y)\, dy$$

$$0 \leq 1 - y \leq x - y \leq x - (x-1) = 1$$

$$\Rightarrow \quad \phi(x-y) = 1$$

$$\Rightarrow \quad (\phi * \phi)(x) = \int_{x-1}^{1} (1)\, dy = 2 - x .$$

Extra part:

We know $\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\omega x} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\omega x}}{-i\omega} \right|_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{\omega} (e^{-i\omega} - 1)$$

By convolution identity,

$$(\phi * \phi)^\wedge(\omega) = \sqrt{2\pi}\,(\hat{\phi}(\omega))^2$$

$$= -\frac{1}{\sqrt{2\pi}}\frac{1}{\omega^2}\left(e^{-i\omega}-1\right)^2$$

$$= -\frac{1}{\sqrt{2\pi}}\frac{1}{\omega^2}\left(e^{-2i\omega}-2e^{-i\omega}+1\right).$$

Butterworth filter:

Show $(Lf)(t) = (h*f)(t)$

$$= \alpha e^{-\alpha t}\int_0^t e^{\alpha u} f(u)\,du$$

Pf: $(Lf)(t) = (h*f)(t)$

$$= \int_{-\infty}^{\infty} h(t-u)\,f(u)\,du$$

$$\overset{f \text{ zero else}}{=} \int_0^{\infty} h(t-u)\,f(u)\,du$$

$$\overset{h \text{ zero else}}{=} \int_0^{t} \underbrace{h(t-u)}_{\alpha e^{-\alpha(t-u)}}\,f(u)\,du$$

$$= \alpha e^{-\alpha t}\int_0^t e^{\alpha u} f(u)\,du$$

Given $f(t) = e^{-\frac{1}{2}t}\left(\sin t + \frac{1}{10}\sin(50t)\right)$

for $t \geq 0$ and $f(t) = 0$ else

we have

$$Lf(t) = \alpha e^{-\alpha t}\left[\int_0^t e^{\alpha u} e^{-\frac{1}{2}u} \sin u \, du\right.$$

$$\left. + \frac{1}{10}\int_0^t e^{\alpha u} e^{-\frac{1}{2}u} \sin(50u)\, du\right]$$

We evaluate for $m = 1$ or $m = 50$

$$\int_0^t e^{(\alpha - \frac{1}{2})u} \underbrace{\sin(mu)}_{\operatorname{Im}(e^{imu})}\, du$$

$$= \operatorname{Im}\left[\int_0^t e^{(\alpha-\frac{1}{2})u + imu}\, du\right]$$

$$= \operatorname{Im}\left[\frac{e^{(\alpha-\frac{1}{2})t + imt} - 1}{\alpha - \frac{1}{2} + im}\right]$$

$$= \operatorname{Im}\left[\frac{(\alpha - \frac{1}{2} - im)\left(e^{(\alpha-\frac{1}{2})t + imt} - 1\right)}{(\alpha-\frac{1}{2})^2 + m^2}\right]$$

$$= \frac{m + e^{(\alpha-\frac{1}{2})t}\left[(\alpha-\frac{1}{2})\sin(mt) - m\cos(mt)\right]}{(\alpha-\frac{1}{2})^2 + m^2}$$

Using this formula gives

$$Lf(t) = \alpha e^{-\alpha t} \left[\frac{e^{(\alpha-\frac{1}{2})t}\left((\alpha-\frac{1}{2})\sin t - \cos(t)\right)+1}{(\alpha-\frac{1}{2})^2+1} \right.$$

$$\left. + \frac{1}{10}\,\frac{e^{(\alpha-\frac{1}{2})t}\left((\alpha-\frac{1}{2})\sin(50t)-50\cos(50t)\right)+50}{(\alpha-\frac{1}{2})^2+50^2} \right]$$

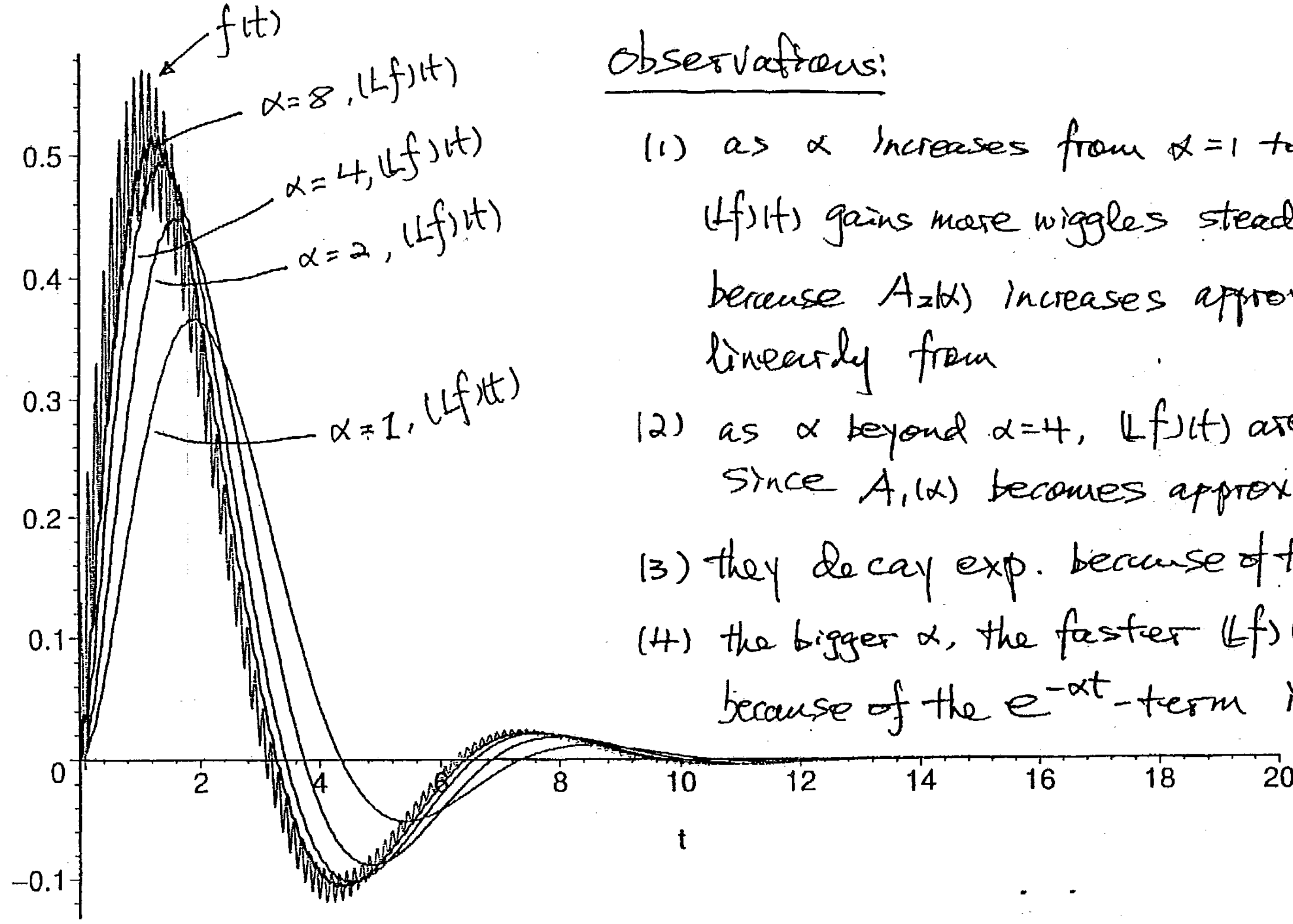
$$= \alpha e^{-\frac{1}{2}t}\left[\frac{(\alpha-\frac{1}{2})\sin t - \cos t}{(\alpha-\frac{1}{2})^2+1} + \frac{1}{10}\,\frac{(\alpha-\frac{1}{2})\sin(50t)-50\cos t}{(\alpha-\frac{1}{2})^2+50^2}\right]$$

$$+ \alpha e^{-\alpha t}\left[\frac{1}{(\alpha-\frac{1}{2})^2+1} + \frac{5}{(\alpha-\frac{1}{2})^2+50^2}\right]$$

$$\overset{\text{trig}}{=} \alpha e^{-\frac{1}{2}t}\left[\underbrace{\frac{1}{\sqrt{(\alpha-\frac{1}{2})^2+1}}}_{A_1(\alpha)}\sin(t-\theta_1) + \underbrace{\frac{1}{10\sqrt{(\alpha-\frac{1}{2})^2+50^2}}}_{A_2(\alpha)}\sin(50t-\theta_2)\right]$$

$$+ \alpha e^{-\alpha t}\underbrace{\left[\frac{1}{(\alpha-\frac{1}{2})^2+1} + \frac{5}{(\alpha-\frac{1}{2})^2+50^2}\right]}_{S(\alpha)}$$

so we want α large to get $A_1(\alpha) \approx \frac{1}{\alpha}$ $A_2(\alpha) \approx 0$ but also $S(\alpha)\,\alpha$ small.



<u>Observations:</u>

(1) as α increases from $\alpha = 1$ to $\alpha = 8$, $(Lf)(t)$ gains more wiggles steadily, because $A_2(\alpha)$ increases approximately linearly from

(2) as α beyond $\alpha = 4$, $(Lf)(t)$ are closer since $A_1(\alpha)$ becomes approximately same $\simeq 1.2$

(3) they decay exp. because of the $e^{-\frac{1}{2}t}$ term

(4) the bigger α, the faster $(Lf)(t)$ decays (slightly) because of the $e^{-\alpha t}$-term in $S_\alpha(t)$.

Matlab script is <u>not</u> included, but students need to hand in.