Math 4355

Assignment 7

1. Take $$m(\omega) = \frac{1}{1+\sqrt{2}\frac{i\omega}{\Omega} - \frac{\omega^2}{\Omega^2}},$$

compute
$$|m(\omega)|^2 = \left[\left(1+\sqrt{2}\frac{i\omega}{\Omega} - \frac{\omega^2}{\Omega^2}\right)\left(1-\sqrt{2}\frac{i\omega}{\Omega} - \frac{\omega^2}{\Omega^2}\right)\right]^{-1}$$
$$= \left[\left(1-\frac{\omega^2}{\Omega^2}\right)^2 + 2\frac{\omega^2}{\Omega^2}\right]^{-1}$$
$$= \frac{1}{1+\frac{\omega^4}{\Omega^4}},$$

Butterworth with $n=4$. Causality below.

To find A, a, b for $h(t)$, note for $\mathrm{Re}\, a,b>0$,

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty A\left(e^{-at} - e^{-bt}\right)e^{-i\omega t}\,dt$$
$$= \frac{A}{\sqrt{2\pi}}\left(\frac{1}{a+i\omega} - \frac{1}{b+i\omega}\right)$$

and
$$\hat{h}(0) = \frac{A}{\sqrt{2\pi}}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{m(0)}{\sqrt{2\pi}}$$

$$\Rightarrow \quad A = \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = \frac{ab}{b-a} .$$

Perform partial fraction decomp on $m(\omega)$,

solve

$$\frac{\omega^2}{\Omega^2} - \sqrt{2}\,\frac{i\omega}{\Omega} - 1 = 0$$

$$\Rightarrow \quad \frac{\omega}{\Omega} = \frac{i}{\sqrt{2}} \pm \sqrt{\frac{1}{2}+1} = \pm\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

so

$$m(\omega) = -\frac{1}{\left(\frac{\omega}{\Omega} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\frac{\omega}{\Omega} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)}$$

$$= \frac{C_1}{\frac{\omega}{\Omega} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} + \frac{C_2}{\frac{\omega}{\Omega} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$$

which gives

$$-1 = C_1\left(\frac{\omega}{\Omega} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + C_2\left(\frac{\omega}{\Omega} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$\Rightarrow \quad 0 = C_1 + C_2 \quad \Rightarrow \quad C_2 = -C_1$$

$$\Rightarrow \quad -1 = 2\,C_1\frac{1}{\sqrt{2}} \quad \Rightarrow \quad C_1 = -\frac{1}{\sqrt{2}} .$$

We have thus

$$m(\omega) = -\frac{1/\sqrt{2}}{\frac{\omega}{\Omega} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} + \frac{1/\sqrt{2}}{\frac{\omega}{\Omega} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$$

$$= -\frac{i\Omega/\sqrt{2}}{i\omega + \Omega\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)} + \frac{i\Omega/\sqrt{2}}{i\omega + \Omega\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$$

so setting $A = \frac{i\Omega}{\sqrt{2}}$ gives

$$a = \Omega\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right), \quad b = \Omega\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

or $A = -\frac{i\Omega}{\sqrt{2}}$ gives

$$a = \Omega\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right), \quad b = \Omega\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

2. Take "digital" Butterworth which replaces $f(\frac{j\pi}{\Omega})$ by

$$\beta_j = \tilde{A} \sum_{k=-\infty}^{j} \left(e^{-a(j-k)\frac{\pi}{\Omega}} - e^{-b(j-k)\frac{\pi}{\Omega}}\right) f\left(\frac{k\pi}{\Omega}\right)$$

This results in

$$g(t) = \tilde{A} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j} \left(e^{-a(j-k)\frac{\pi}{\Omega}} - e^{-b(j-k)\frac{\pi}{\Omega}}\right) f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}$$

$$\overset{k\to k-j}{=} \tilde{A} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{0} \left(e^{+ak\frac{\pi}{\Omega}} - e^{+bk\frac{\pi}{\Omega}}\right) f\left(\frac{(k+j)\pi}{\Omega}\right) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}$$

$$\overset{j\to j-k}{=} \tilde{A} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{0} \left(e^{+ak\frac{\pi}{\Omega}} - e^{+bk\frac{\pi}{\Omega}}\right) f\left(\frac{j\pi}{\Omega}\right) \frac{\sin(\Omega(t+\frac{k\pi}{\Omega}) - j\pi)}{\Omega(t+\frac{k\pi}{\Omega}) - j\pi}$$

$$= \tilde{A} \sum_{k=-\infty}^{0} f\left(t + k\frac{\pi}{\Omega}\right)\left(e^{+ak\frac{\pi}{\Omega}} - e^{+bk\frac{\pi}{\Omega}}\right)$$

Fourier transform

$$\hat{g}(\omega) = \tilde{A} \sum_{k=0}^{\infty} e^{-ki\pi\omega/\Omega} \left(e^{-ak\frac{\pi}{\Omega}} - e^{-bk\frac{\pi}{\Omega}}\right) \hat{f}(\omega)$$

So system function is

$$m(\omega) = \tilde{A} \sum_{k=0}^{\infty} \left(e^{-k\pi(i\omega + a)/\Omega} - e^{-k\pi(i\omega+b)/\Omega} \right)$$

$$\overset{\text{geom.}}{\underset{\text{ser.}}{=}} \tilde{A} \left(\frac{1}{1 - e^{-\pi(i\omega+a)/\Omega}} - \frac{1}{1-e^{-\pi(i\omega+b)/\Omega}} \right)$$

To get $m(0) = 1$,

$$1 = \tilde{A} \left(\frac{1}{1 - e^{-\pi a/\Omega}} - \frac{1}{1 - e^{-\pi b/\Omega}} \right)$$

$$\Rightarrow \quad \tilde{A}^{-1} = \frac{(\cancel{1} - e^{-\pi b/\Omega}) - (\cancel{1} - e^{-\pi a/\Omega})}{(1 - e^{-\pi a/\Omega})(1 - e^{-\pi b/\Omega})}$$

3. Choosing $a = \Omega\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$, $b = \Omega\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

gives

$$\tilde{A}^{-1} = \frac{e^{-\pi\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} - e^{-\pi\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)}}{1 + 2i e^{-\pi/\sqrt{2}} \sin\left(\frac{\pi}{\sqrt{2}}\right) + e^{-\sqrt{2}\pi}}$$

$$\tilde{A} = -\frac{1 + 2i e^{-\pi/\sqrt{2}} \sin\left(\frac{\pi}{\sqrt{2}}\right) + e^{-\sqrt{2}\pi}}{2i e^{-\pi/\sqrt{2}} \sin\left(\frac{\pi}{\sqrt{2}}\right)}$$