UNIVERSITY OF HOUSTON
PRACTICE FINAL

Term: Spring **Year:** 2009

Student: First Name ____________________ **Last Name** ____________________

UH Student ID Number ____

Course Abbreviation and Number	Math 4355
Course Title	Mathematics of Signal Representations
Section(s)	001
Sections of Combined Course(s)	
Section Numbers of Combined Course(s)	
Instructor(s)	B. G. Bodmann

Date of Exam	A day in May, 2009
Time Period	Start time: 5:30pm End time: 8:00pm
Duration of Exam	2 1/2 hours
Number of Exam Pages (including this cover sheet)	14 pages
Exam Type	Closed book
Additional Materials Allowed	None

Grading Scheme:

Question	**Points**	**Score**
1	25	
2	15	
3	35	
4	20	
5	25	
Total	120	

1. (25 points) Fourier series. Consider the function $f(x) = |\sin(x/2)|$ on $x \in [-\pi, \pi)$.

   (a) (10 points) Is the Fourier series of f pointwise convergent to $f(x)$ at each x in the interval $[-\pi, \pi)$? Quote conditions from an appropriate theorem to justify your answer.

   (b) (5 points) Sketch three periods of the function to which the Fourier series converges.

(c) (10 points) Find all the Fourier coefficients a_n and b_n (real form of the Fourier series) for f.

2. Let $f(x) = e^{-|x|}$.

   (a) (5 points) State Plancherel's theorem.

   (b) (5 points) Compute the Fourier transform, $\hat{f}$, of f.

(c) (5 points) Use Plancherel's theorem to compute the value

$$A = \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} d\omega \,.$$

3. (35 points) Fourier transform and filters. You are given two (causal) Butterworth filters L_1 and L_2, with impulse responses $h_1(t) = 3e^{-t}$ and $h_2(t) = \frac{2}{3}e^{-2t}$ for $t \geq 0$ and $h_1(t) = h_2(t) = 0$ for $t < 0$, respectively. By chaining them together, you obtain a filter L which has impulse response $h_1 * h_2$.

   (a) (10 points) Compute $h(t) = (h_1 * h_2)(t)$. Distinguish cases, if necessary. Is L a causal filter?

(b) (15 points) Compute the system function m of L. You may use properties of the Fourier transform on page 13 in your calculation.

(c) (10 points) State the properties that a system function m of a low-pass filter satisfies. Is L a low-pass filter?

4. (20 points) Norm and sampling theorem.

   (a) (5 points) State a theorem relating the norm of $f \in L^2(\mathbb{R})$ to that of its Fourier transform $\hat{f}$.

   (b) (5 points) State a theorem relating the norm of a function $g \in L^2([-\pi, \pi])$ and the norm of its (complex) Fourier coefficients $\{\alpha_k\}_{k \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$.

(c) (10 points) Let f be a π-bandlimited ($\Omega = \pi$), continuous square-integrable function with sample values

$$f(k) = \begin{cases} \sqrt{k}, & 0 \leq k \leq 100 \\ 0, & \text{else} \end{cases} .$$

Recall that the Fourier transform $\hat{f}$ of such a function satisfies $\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega}$ when $\omega \in [-\pi, \pi]$, with $c_k = \frac{1}{\sqrt{2\pi}} f(-k)$. Use the preceding parts of this problem to calculate the squared norm $\|f\|^2$.

5. (25 points) The Haar wavelet. Let f be the piecewise defined function in $L^2(\mathbb{R})$,

$$f(x) = \begin{cases} \sqrt{x}, & 0 \leq x < 1 \\ 0, & \text{else} \end{cases} .$$

Let ϕ be the Haar scaling function, $\phi(x) = 1$ when $x \in [0, 1)$ and $\phi(x) = 0$ otherwise, and let

$$\psi(x) = \begin{cases} 1, & 0 \leq x < 1/2 \\ -1, & 1/2 \leq x < 1 \\ 0, & \text{else} \end{cases} .$$

(a) (5 points) State a (general) formula for f_3, the orthogonal projection of f onto V_3, in terms of the orthonormal basis vectors $\phi_k(x) = \phi(x - k)$ and $\psi_k^{(j)}(x) = 2^{j/2}\psi(2^j x - k)$ for suitable values of j and k.

(b) (5 points) At $x = 0$, the wavelet decomposition results in $f_3(0) = f_0(0) + \sum_{j=0}^{2} w_j(0)$, with $f_0 \in V_0$ and $w_j \in W_j \equiv V_j^{\perp} \cap V_{j+1}$. Compute $f_0(0)$ and $w_j(0)$, $0 \leq j \leq 2$.

(c) (10 points) Each function in V_2 is determined by its function values $\{f(2^{-2}k)\}_{k\in\mathbb{Z}}$, similar to the sampling theorem for band-limited functions. Find an expression for $f(t)$, $t \in \mathbb{R}$, in terms of $\{f(2^{-2}k)\}_{k\in\mathbb{Z}}$. Distinguish cases. You may use a suitable orthonormal basis for V_2 to obtain your answer.

(d) (5 points) Check your answer for part (b) of this problem with the sampling theorem for V_2 from part (c).

Essentials of the Fourier Transform

1. $\hat{f}(\lambda) = \mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\lambda}\, dx$
2. $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{ix\lambda}\, d\lambda$
3. $\mathcal{F}[x^n f(x)](\lambda) = i^n \hat{f}^{(n)}(\lambda)$
4. $\mathcal{F}[f^{(n)}(x)](\lambda) = (i\lambda)^n \hat{f}(\lambda)$
5. $\mathcal{F}[f(x-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda)$
6. $\mathcal{F}[f(bx)](\lambda) = \frac{1}{b} \hat{f}(\frac{\lambda}{b})$
7. $\mathcal{F}[f * g](\lambda) = \sqrt{2\pi} \hat{f}(\lambda) \hat{g}(\lambda)$

Trigonometric formulas

1. $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta,$
2. $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta\,.$

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