Exam 2 – Math 4377
November 20, Fall 2008

Show all the work for full credit. In this exam no calculators are allowed. Answer all of the following questions. The maximum score is 150.

1. (15 Points) Let $V = \mathbb{R}^2$ and W be the vector space of all real 2×2 matrices. Define a linear transformation $T : V \to W$ by
$$T(x, y) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix}.$$
Show that T is one-to-one. You may quote results from class to simplify your answer.

2. (25 points) Isomorphisms:
   (a) Define what it means for two vector spaces V and W to be isomorphic.
   (b) Prove that if $\dim(V) = n$ and $\dim(W) = m$ and V and W are isomorphic vector spaces, then $m = n$. You may quote an appropriate property of isomorphisms to simplify your answer.
   (c) Prove that if $T : V \to V$ is linear and V is a finite dimensional vector space, and T is onto, then it is invertible. You may quote results from class (other than this theorem) to simplify your proof.

3. (30 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by
$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$
   (a) Compute the matrix representation $[T]_E^E$ with respect to the standard (ordered) basis $E = \{\epsilon_1, \epsilon_2, \epsilon_3\}$.
   (b) Given the ordered basis $B = \{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$, then $[T]_B^B$ has an expression in the form $[T]_B^B = QAQ^{-1}$. Specify the matrix A and compute the matrices Q and Q^{-1} in this expression. There is no need to compute the matrix product QAQ^{-1}!

4. (30 points) Let V be a vector space over the field of real numbers and suppose T is an isomorphism of V onto $\mathbb{R}^3$. Let α_1, α_2, α_3 and α_4 be vectors in V such that $T(\alpha_1) = (\sqrt{2}, \sqrt{2}, -\sqrt{2})$, $T(\alpha_2) = (2 + \sqrt{2}, \sqrt{2}, 2 - \sqrt{2})$, $T(\alpha_3) = (1 + \sqrt{2}, 1, \sqrt{2} - 1)$, and $T(\alpha_4) = (1, \sqrt{2}, \sqrt{2})$.
   (a) Is the set $\{\alpha_1, \alpha_2, \alpha_3\}$ linearly independent? Explain why/why not.
   (b) Is α_4 in the span of $\{\alpha_1, \alpha_2, \alpha_3\}$? Explain why/why not.

5. (25 points) Find the dual basis $B^* = \{f_1, f_2, f_3\}$ for the basis $B = \{(1,0,1),\ (0,1,2),\ (-1,-1,0)\}$ of $\mathbb{R}^3$. Each vector f in the dual basis should be written in the form

$$f(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3$$

with a choice of coefficients $c_1,\ c_2,\ c_3 \in \mathbb{R}$.

6. (25 points) Let V be the vector space of polynomials over $\mathbb{R}$, let $x_1 < x_2 \in \mathbb{R}$ be given and define for any pair of polynomials $p, q \in V$ the function $D(p,q) = p(x_1)q(x_2) - p(x_2)q(x_1)$.

   (a) Verify that D is linear in the first entry.

   (b) Verify that D is alternating.

   (c) Using the preceding parts, verify that D is 2-linear.

Math 4377

Exam 2

1. Since T is linear, to show it is 1-1 is equivalent to $\ker(T) = \{0\}$.

So assume $T(x,y) = \begin{pmatrix} x+7y & 5y \\ -10y & x-7y \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $-10y = 0 \Rightarrow y = 0$

$\Rightarrow x + 7y = x = 0$

$\Rightarrow (x,y) = (0,0)$.

2. a) Two vector spaces V, W are isomorphic if there is an isomorphism $T: V \to W$ (i.e. a bijection, or a 1-1 and onto map).

b) Since an isomorphism T maps bases to bases, given $\mathcal{B} = \{\beta_1, \beta_2, \dots \beta_n\}$ basis of V, we know $\{T(\beta_1), T(\beta_2), \dots T(\beta_n)\}$ is basis for W.

Since $\dim(W) = m = \#$ basis vectors

for any basis, $m = n$.

c) If $T: V \to V$ is onto, then $\operatorname{ran}(T) = V$, $\operatorname{rank}(T) = \dim(V)$. By rank - nullity,

$$\dim(\ker(T)) + \underbrace{\dim(\operatorname{ran}(T))}_{\dim(V)} = \dim(V)$$

$\Rightarrow \dim(\ker(T)) = \operatorname{nullity}(T) = 0$.

$\Rightarrow \ker(T) = \{0\} \Rightarrow T$ 1-1 and onto

$\Rightarrow T$ invertible. $\}$ 2

3. a) $T(1,0,0) = (3,-2,-1)$

$T(0,1,0) = (0,1,2)$

$T(0,0,1) = (1,0,4)$

$$\Rightarrow [T]_E = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}.$$

b) $[T]_B = [id]_E^B \, [T]_E^E \, [id]_B^E$

where $[id]_B^E = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

and $[id]_E^B = \left([id]_B^E\right)^{-1}$

Compute inverse

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5/2 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & -\frac{3}{4} & \frac{5}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & +\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\Rightarrow \quad [T]_{\mathcal{B}} = \underbrace{\begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} & \frac{5}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{Q^{-1}}$$

4. a) $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly indep.

$\iff \{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly indep.

b/c T is isomorphism

To test whether $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly indep., row reduce

$$\begin{pmatrix} 2\sqrt{2} & 2+\sqrt{2} & 1+\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 1 \\ -\sqrt{2} & 2+\sqrt{2} & \sqrt{2}-1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \sqrt{2} & 2+\sqrt{2} & 1+\sqrt{2} \\ 0 & -2 & -\sqrt{2} \\ 0 & 4 & 2\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} \sqrt{2} & 2+\sqrt{2} & 1+\sqrt{2} \\ 0 & -2 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ linearly dep. !

b) Since third variable is free, can remove $T(\alpha_3)$ and get basis $\{T(\alpha_1), T(\alpha_2)\}$ for span of $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$.

Test $\{T(\alpha_1), T(\alpha_2), T(\alpha_4)\}$

$$\begin{pmatrix} \sqrt{2} & 2+\sqrt{2} & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 2-\sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} \sqrt{2} & 2+\sqrt{2} & 1 \\ 0 & -2 & \sqrt{2}-1 \\ 0 & 4 & 1+\sqrt{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} \sqrt{2} & 2+\sqrt{2} & 1 \\ 0 & -2 & \sqrt{2}-1 \\ 0 & 0 & 3\sqrt{2}-1 \end{pmatrix}$$

$\Rightarrow$ linearly indep. ?

$\Rightarrow$ $\alpha_4 \notin \operatorname{span}\{\alpha_1, \alpha_2\}$.

5. We want matrix C s.th. $CB = I$

so invert matrix with basis vectors as columns

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right)$$

then coefficients are

$$f_1(x_1, x_2, x_3) = \frac{2}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3$$

$$f_2(x_1, x_2, x_3) = -\frac{1}{3}x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3$$

$$f_3(x_1, x_2, x_3) = -\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3$$

6. a) $D(p_1 + cp_2, q)$ $\quad$ (p_1, p_2 poly, $c \in \mathbb{R}$)

$$= (p_1(x_1) + cp_2(x_1))\, q(x_2) - (p_1(x_2) + cp_2(x_2))\, q(x_1)$$

$$= p_1(x_1) q(x_2) - p_1(x_2) q(x_1) + c\,(p_2(x_1) q(x_2) - p_2(x_2) q(x_1))$$

$$= D(p_1, q) + c\, D(p_2, q)$$

b) $D(p, q) = p(x_1) q(x_2) - p(x_2) q(x_1)$

$$= -(q(x_1) p(x_2) - q(x_2) p(x_1))$$

$$= -D(q, p)$$

(over $\mathbb{R}$, this is equivalent to $D(p,p) = 0$ for all p poly)

c) To verify linearity in 2nd entry,

$D(p, q_1 + cq_2)$

$$\overset{b)}{=} -D(q_1 + cq_2, p)$$

$$\overset{a)}{=} -D(q_1, p) - c\, D(q_2, p)$$

$$\overset{b)}{=} D(p, q_1) + c\, D(p, q_2)$$