Math 4377

Homework Set 10

3.6 1a) $W = \{(x_1, x_2, \dots x_n) : x_1 + x_2 + \dots + x_n = 0\}$

$W^\circ = \{ f \in V^* : \underbrace{f(x_1, x_2, \dots x_n)}_{c_1 x_1 + c_2 x_2 + \dots + c_n x_n = 0} = 0 \ \forall \ x \in W\}$

We know $f(x_1, \dots x_n) = x_1 + x_2 + \dots + x_n$ is in W_0, and $\dim W + \dim W^\circ = n$ so by $\dim W = n-1$, $\dim W^\circ = 1$ and thus $\{f\}$ is a basis for W°. Thus, every $g \in W^\circ$ is $g = cf$ for some $c \in F$.

b) By $\dim(W) = \dim(W^*)$ and $\dim(W) = n-1$, we only have to find a basis for W^* consisting of functionals $\{f_j\}_{j=1}^{n-1}$ such that for each j,

$$f_j(x_1, \dots x_n) = c_1^{(j)} x_1 + \dots + c_n^{(j)} x_n$$

and $c_1^{(j)} + c_2^{(j)} + \dots + c_n^{(j)} = 0$. Moreover, $n-1$ linear indep. functionals are enough.

Given a basis $\{f_j\}_{j=1}^{n-1}$ for W^*,

$$f_j(x_1, \ldots x_n) = c_1^{(j)} x_1 + \ldots + c_n^{(j)} x_n$$

let

$$d_j = c_1^{(j)} + c_2^{(j)} + \ldots + c_n^{(j)}.$$

Now consider

$$\tilde{f}_j(x_1 \ldots x_n) = (\underbrace{c_1^{(j)} - \frac{d_j}{n}}_{\tilde{c}_1^{(j)}}) x_1 + \ldots + (\underbrace{c_n^{(j)} - \frac{d_j}{n}}_{\tilde{c}_n^{(j)}}) x_n$$

then

$$\tilde{c}_1^{(j)} + \tilde{c}_2^{(j)} + \ldots + \tilde{c}_n^{(j)} = d_j - d_j = 0.$$

Moreover, for $x \in W$, $f_j(x) = \tilde{f}_j(x)$.

Since $\dim \{(\hat{c}_1, \hat{c}_2, \ldots \hat{c}_n) : \hat{c}_1 + \hat{c}_2 + \ldots + \hat{c}_n = 0\}$

$= n-1$

the $\tilde{c}_j$ are unique.

3.7 | 1 a) $g(x_1, x_2) = (f \circ T)(x_1, x_2)$

$= f(T(x_1, x_2))$

$= f(x_1, 0) = ax_1$

b) $g(x_1, x_2) = f(T(x_1, x_2))$

$= f(-x_2, x_1)$

$= -ax_2 + bx_1$

c) $g(x_1, x_2) = f(T(x_1, x_2))$

$= f(x_1 - x_2, x_1 + x_2)$

$= a(x_1 - x_2) + b(x_1 + x_2)$

$= (a+b)x_1 + (b-a)x_2$

6 Let $D^t : P_n(\mathbb{R})^* \to P_n(\mathbb{R})^*$

by $D^t(g) = g \circ D$ for all $g \in P_n(\mathbb{R})^*$.

So $\ker(D^t) = \{ g \in P_n(\mathbb{R})^* : g(p') = 0$
$\forall\ p \in P_n(\mathbb{R}) \}$

But $\operatorname{ran}(D) = P_{n-1}(\mathbb{R})$, so

$g(c_0 + c_1 x + c_2 x^2 + \dots + c_{n-1} x^{n-1}) = 0$

for all c_j's, or $g \in (\operatorname{ran}(D))^0$

as proved in class. Now, $\dim(\operatorname{ran}(D))^0$

$+ \underbrace{\dim(\operatorname{ran}(D))}_{n-1} = n \Rightarrow \dim(\operatorname{ran}(D))^0 = 1.$

We only need to find $\underline{\text{one}}$ functional.

Given the standard basis $\{1, x, x^2, \dots x^n\}$ and its dual $\{f_1, f_2, \dots f_n\}$, this is, e.g. f_n, b/c $f_n(x^j) = 0 \quad \forall j \leq n-1$. So $\{f_n\}$ is a basis for $\ker(D^t)$.

5.2 1 a) Let $A_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $A_2 = A_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $c \neq 1$

then $D(cA_1, A_2, A_3) = c+1$

$\neq cD(A_1, A_2, A_3) = c(1+1) = 2c$.

Not linear in $A_1 \Rightarrow$ not 3-linear.

b) Same A_1, A_2, A_3 as in a), $c^2 - c \neq 0$

$$D(cA_1, A_2, A_3) = c^2 + 3c$$

$$\neq cD(A_1, A_2, A_3) = c(1+3)$$

Not 3-linear.

c) $D(A_1, A_2, A_3) = A_{11} A_{22} A_{33}$

Linearity in A_1:

$D(0, A_2, A_3) = 0$ ✓

$D(cA_1 + B_1, A_2, A_3)$

$= (cA_{11} + B_{11}) A_{22} A_{33}$

$= cA_{11} A_{22} A_{33} + B_{11} A_{22} A_{33}$

$= cD(A_1, A_2, A_3) + D(B_1, A_2, A_3)$

similar for $A_2, A_3 \Rightarrow$ 3-linear.

d) Pick $A_1 = A_2 = A_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then

$$D(A_1, cA_2, A_3) = (1)(c)(c) + 5(c)(c)(c)$$

$$= c^2 + 5c^3 .$$

$$\neq cD(A_1, A_2, A_3) = 6c \quad \text{for e.g. } c = \frac{1}{2}$$

b/c $\frac{1}{4} + \frac{5}{8} = \frac{7}{8} < \frac{6}{2} = 3$.