Math 4377

Homework #11

1. a) Characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= (1-\lambda)(-\lambda)$$

so eigenvalues are $\lambda = 1$, $\lambda = 0$ over $\mathbb{C}$ and $\mathbb{R}$ alike, meaning for $\mathcal{U}$ and for T.

Eigenvectors are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, because

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad (\lambda = 1)$$

and

$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad (\lambda = 0)$$

Since $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ forms a basis for $\mathbb{R}^2$ or for $\mathbb{C}^2$, these vectors span the corresp. eigenspaces

$$E_1 = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix}, \quad E_2 = \mathbb{R}\begin{pmatrix}0\\1\end{pmatrix}$$

over $\mathbb{R}$ and

$$E_1 = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}, \qquad E_2 = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$$

over $\mathbb{C}$.

b) Characteristic polynomial

$$p_A(\lambda) = (2-\lambda)(1-\lambda)+3$$
$$= 5 - 3\lambda + \lambda^2$$

so over $\mathbb{R}$

$$p_A(\lambda) = \underbrace{\lambda^2 - 3\lambda + \frac{9}{4}}_{(\lambda-\frac{3}{2})^2 \geq 0} + \frac{11}{4} \geq \frac{11}{4} \neq 0$$

no solution, i.e. no eigenvalues.

Over $\mathbb{C}$,

$$p_A(\lambda) = \left(\lambda - \tfrac{1}{2}(3-i\sqrt{11})\right)\left(\lambda - \tfrac{1}{2}(3+i\sqrt{11})\right)$$

and we obtain eigenspaces

for $\ker\left(A - \frac{1}{2}(3-i\sqrt{11})I\right)$, solve

$$\begin{pmatrix} 2-\frac{1}{2}(3-i\sqrt{11}) & 3 \\ -1 & 1-\frac{1}{2}(3-i\sqrt{11}) \end{pmatrix}\vec{x} = 0$$

Reduce coeff. matrix

$$\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{11}}{2} i & 3 \\ -1 & -\frac{1}{2} + \frac{\sqrt{11}}{2} i \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{11}}{2} i & 3 \\ -\frac{1}{2} - \frac{\sqrt{11}}{2} i & \underbrace{-\frac{1}{4} - \frac{11}{4}}_{-3} \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{11}}{2} i & 3 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \left(\frac{1}{2} + \frac{\sqrt{11}}{2} i\right) x_1 + 3 \underset{\substack{\uparrow \\ \text{free}}}{x_2} = 0$$

choosing $x_2 = 1$ gives

$$x_1 = -\frac{1}{2} + \frac{\sqrt{11}}{2} i$$

$$\Rightarrow \quad E_{\frac{1}{2} - \frac{\sqrt{11}}{2} i} = \mathbb{C} \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{11}}{2} i \\ 1 \end{pmatrix}$$

Similarly, $\ker\left(A - \frac{1}{2}(3+i\sqrt{11})I\right)$

$$= E_{\frac{1}{2}(3+i\sqrt{11})} = \mathbb{C}\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{11}}{2}i \\ 1 \end{pmatrix}.$$

4. Char polynomial

$$p_A(\lambda) = -\lambda^3 + \lambda^2 + 5\lambda + 3$$

$$= -(\lambda-3)(\lambda+1)^2$$

$\Rightarrow$ $\lambda = 3$ is eigenvalue

Eigenvectors in E_3

$$\begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

coeff matrix

$$\begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix} \sim \begin{pmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ -4 & 2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow$
free

so eigenvectors are

$$E_3 = \mathbb{R}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

For $\lambda = -1$, we get

$$\begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ x_2, x_3 free variables

$$E_{-1} = \operatorname{span}\left\{\begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}\right\}.$$

10. Char polynomial

$$p_T = \det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}$$

$$= (a-\lambda)(c-\lambda) - b^2.$$

Case I: $b \neq 0$, assume WLOG $a < c$

then for $\lambda = a$ or $\lambda = c$

$$p_T(\lambda) = -b^2 < 0$$

but $p_T(\lambda) = \lambda^2 - (a+c)\lambda + ac - b^2$

so $p_T(\lambda) \to +\infty$ as $\lambda \to \pm\infty$

and by continuity $\exists\ \lambda_1 < a$

$\lambda_2 > \cancel{b}c$ s.th. $p_T(\lambda_1) = p_T(\lambda_2) = 0$.

Since there are two distinct eigenvalues, we have two lin. indep. eigenvectors, i.e. a basis for $\mathbb{R}^2$ of eigenvectors.

Hence T is diagonalizable

Case II: $b = 0$, so T is diagonal in standard basis E,

$$[T]_E = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$