MATH 4377, SECTION 33224
HOMEWORK
DUE TUESDAY, SEPTEMBER 16th

1. Let A be an mxm matrix over a field F and suppose there is an mxm matrix B such that $BA = I_m$. Prove that A is invertible and that $B = A^{-1}$

2. Let $V = R^{1x2}$, the set of all 1x2 matrices with real entries. Define:

   vector addition: $[x_1 \; x_2] + [y_1 \; y_2] \equiv [x_1 + y_1 \;\; 0]$

   scalar multiplication: $c \; [x_1 \; x_2] \equiv [cx_1 \; cx_2]$ where $c \in R$.

   Which of the axioms of a vector space are satisfied and which are not. Justify your answers.

3. Let V be a vector space over the field F, $\alpha \in V$, $c \in F$.
   Show that if $c\,\alpha = \theta$ then either $c = 0$ or $\alpha = \theta$.

4. Let V be vector space over the reals consisting of all continuous real-valued functions defined on the closed interval [0,1], with the usual operations of addition and scalar multiplication of functions. Let W consist of all functions f (t) in V for which $\int_0^1 f(t)\,dt = 0$.

   Show that W is a subspace of V.

Math 4377

Homework due Sep 16, 2008

1. Let A, B be $m \times m$ matrices, and assume $BA = I_m$, then for any $m \times m$ matrix X, $AX = 0$ implies $X = 0$ b/c $AX = 0 \Rightarrow BAX = 0 \Rightarrow X = 0$. Now, by Thm 13, we know A is invertible. } Step 1

Moreover, the right inverse, C, is identical to B, because

$$BA = I_m = AC$$

$$\Rightarrow \quad B = BI_m = BAC = I_mC = C.$$

} Step 2

2. Axioms:

- addition $[x_1 \ x_2] + [y_1 \ y_2] = [x_1 + y_1 \ \ 0] = [y_1 + x_1 \ \ 0] = [y_1 \ y_2] + [x_1 \ x_2]$ ✓

commutative!

- addition $[x_1 \ x_2] + ([y_1 \ y_2] + [z_1 \ z_2])$
$= [x_1 \ x_2] + [y_1 + z_1 \ \ 0]$
$= [x_1 + y_1 + z_1 \ \ 0]$
$= [x_1 + y_1 \ \ 0] + [z_1 \ z_2]$
$= ([x_1 \ x_2] + [y_1 \ y_2]) + [z_1 \ z_2]$ ✓

associative

- unique zero vector

$$[x_1 \quad x_2] + [y_1 \quad y_2] = [x_1 + y_1 \quad 0]$$

$$\neq [x_1 \quad x_2]$$

unless $x_2 = 0$, existence fails!

- unique negative

$$[x_1 \quad x_2] + [y_1 \quad y_2] = [x_1 + y_1 \quad 0]$$

$$= [0 \quad 0]$$

$$\Rightarrow \begin{cases} y_1 = -x \\ y_2 \text{ can be anything} \end{cases}$$

uniqueness fails!

- scalar multiplication

$$c[x_1 \quad x_2] = [cx_1 \quad cx_2]$$

has desired properties b/c is as in lecture.

3. Let $\alpha \in V$, $c \in F$ and $c\alpha = 0$.

Assume both c and α are non-zero,

then $\frac{1}{c}(c\alpha) = \alpha$, but also

$$\frac{1}{c}(c\alpha) = \frac{1}{c}(0) = 0$$

which is a contradiction. Consequently,

~~either~~ $c = 0$ or $\alpha = 0$.

4. We check two properties:

- $f(t) = 0$ is in V, b/c

$$\int_0^1 f(t)\,dt = \int_0^1 0\,dt = 0$$

- If $f, g \in V$ and $c \in \mathbb{R}$, then so is $f + cg$, because

$$\int_0^1 (f(t) + c\,g(t))\,dt$$

$$= \int_0^1 f(t)\,dt + c\int_0^1 g(t)\,dt$$

$$= 0 + c(0) = 0.$$

alt.: check
$f \in V \Rightarrow cf \in V$
$f, g \in V \Rightarrow f + g \in V$

Department of Mathematics University of Houston

Math 4377

Advanced Linear Algebra

Fall 2008

Homework Set 5, due Thursday, Sep 25, 1pm

Section 2.2

1 Which of the following sets of vectors $\alpha = (a_1, a_2, \dots a_n) \in \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ ($n \geq 3$)? If yes, show it has the properties required of a subspace; if not, find an example that shows how a property of subspaces is violated:

(a) all α such that $a_1 \geq 0$;

(b) all α such that $a_1 + 3a_2 = a_3$;

(c) all α such that $a_2 = a_1^2$;

(d) all α such that $a_1 a_2 = 0$;

(e) all α such that a_2 is rational.

4 Let W be the set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfy

$$\begin{array}{rcrcrcrcrl} 2x_1 & - & x_2 & + & \frac{4}{3}x_3 & - & x_4 & & & = 0 \\ x_1 & & & & +\frac{2}{3}x_3 & & & - & x_5 & = 0 \\ 9x_1 & - & 3x_2 & + & 6x_3 & - & 3x_4 & - & 3x_5 & = 0. \end{array}$$

Find a finite set of vectors which spans W.

5 Let F be a field and let n be an integer with $n \geq 2$. Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A are subspaces of V? If yes, show it has the properties required of a subspace; if not, show how a property of subspaces is violated:

(a) all invertible A;

(b) all non-invertible A;

(c) all A such that $AB = BA$ where B is a fixed matrix in V;

(d) all A such that $A^2 = A$.

Section 2.3

2 Are the vectors $\alpha_1 = (1, 1, 2, 4)$, $\alpha_2 = (2, -1, -5, 2)$, $\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$ linearly independent in $\mathbb{R}^4$?

3 Find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors α_1, α_2, α_3 and α_4 in the preceding exercise.

4 Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, and $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors ϵ_1, ϵ_2 and ϵ_3 as a linear combination of α_1, α_2, and α_3.

Math 4377

Homework Set 5,

due Sept. 25, 2008

2.2 1. a) $S = \{(a_1, \dots a_n) : a_1 \geq 0\}$

is <u>not</u> a subspace, b/c

$(a_1, 0, \dots 0) \in S$ for $a_1 \geq 0$

but not

$-(a_1, 0, \dots 0) = (\underbrace{-a_1}_{<0}, 0, \dots 0)$.

b) $S = \{(a_1, \dots a_n) : a_1 + 3a_2 = a_3\}$

i) $(0, 0, \dots 0) \in S$ ✓

ii) if $\alpha, \beta \in S$ then for $c \in F$

$$a_1 + 3a_2 = a_3$$
$$cb_1 + 3cb_2 = cb_3$$

so

$$(a_1 + cb_1) + 3(a_2 + cb_2) = a_3 + cb_3$$

which means

$$(a_1 + cb_1, a_2 + cb_2, \dots) \in S$$

$$\alpha + c\beta \in S.$$

So S is a subspace

c) $S = \{(a_1, \dots a_n) : a_2 = a_1^2\}$

We have $(1, 1, 0, \dots 0) \in S$

but not $(c, c, 0, \dots 0)$

for $c \notin \{0, 1\}$, b/c then $c \neq c^2$,

so S is not a subspace.

d) $S = \{(a_1, a_2, \dots a_n) : a_1 a_2 = 0\}$

We have $(1, 0, 0, \dots 0) \in S$

and $(0, 1, 0, \dots) \in S$

but $(1, 1, 0, \dots) \notin S$

$= (1, 0, \dots 0)$

$+ (0, 1, \dots 0)$,

so S is not a subspace

e) $S = \{(a_1, a_2, \dots a_n) : a_2 \in \mathbb{Q}\}$

We have $(0, 1, 0, \dots 0) \in S$

but $(0, \pi, 0, \dots 0) \notin S$

$\parallel$

$\pi(0, 1, 0, \dots 0)$

so S is not a subspace.

4. Let $(x_1, x_2, \ldots x_5) \in \mathbb{R}^5$ satisfy

$$2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \tfrac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Row reduce

$$\begin{pmatrix} 2 & -1 & 4/3 & 0 & -1 \\ 1 & 0 & 2/3 & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2/3 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so after choosing x_3, x_4, x_5, we have determinded values of x_1, x_2

$$x_1 = -\tfrac{2}{3}x_3 + x_5$$

$$x_2 = -x_4 + 2x_5.$$

Spanning set is

$$\{(-\tfrac{2}{3}, 0, 1, 0, 0), (0, -1, 0, 1, 0), (1, 2, 0, 0, 1)\}.$$

5. a) $S = \{ A \in F^{n \times n} : A \text{ invertible} \}$

is not a vector space b/c

O (zero-matrix) is not included

b) $S = \{ A \in F^{n \times n} : A \text{ non-invertible} \}$

Let

$$D_1 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{pmatrix} \Big\}\, n-1$$

$$D_2 = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & 1 \end{pmatrix} \Big\}\, n-1$$

then

D_1, D_2 are not invertible

but $D_1 + D_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is !

Not closed under addition.

c) $S = \{ A \in F^{n \times n} : AB = BA \}$

for fixed B

i) 0 matrix is in S

ii) given $A_1, A_2 \in S$

we know $A_1 B = B A_1$

$A_2 B = B A_2$,

so

$$(A_1 + cA_2) B$$

$$= A_1 B + c A_2 B$$

$$= B A_1 + c B A_2$$

$$= B(A_1 + cA_2)$$

which means $A_1 + cA_2 \in S$

$\Rightarrow$ S is a subspace!

d) $S = \{ A \in F^{n \times n} : A^2 = A \}$

$$D = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in S$$

but

$$2D = \begin{pmatrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{pmatrix}$$ gives $(2D)^2 = 4D$

$= 2(2D)$

So $2D \notin S$ $\Rightarrow$ not a subspace.

2.3 2. $\alpha_1 = (1, 1, 2; 4)$, $\alpha_2 = (2, -1, -5, 2)$

$\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \begin{matrix} \\ \ominus \\ \\ \ominus \end{matrix} \sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & 2 & 6 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

does not have full rank, so

vectors are linearly dependent.

3. Continue eliminating rows

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so basis vectors are

$$B = \{(1, 1, 2, 4), (0, 3, 9, 6)\}$$

4. Vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$,
$\alpha_3 = (0,-3,2)$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & -3 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

has maximal rank $\Rightarrow$ $\{\alpha_1, \alpha_2, \alpha_3\}$ lin. indep.

Want to find $(P_{ij})_{i,j=1}^{3}$

so that $\varepsilon_j = \sum_{i=1}^{3} P_{ij}\, \alpha_i$.

We get

$$P = \begin{pmatrix} \frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

so

$$\varepsilon_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3$$

$$\varepsilon_2 = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3$$

$$\varepsilon_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 .$$