Department of Mathematics University of Houston

Math 4377

Advanced Linear Algebra

Fall 2008

Homework Set 5, due Thursday, Sep 25, 1pm

Section 2.2

1 Which of the following sets of vectors $\alpha = (a_1, a_2, \ldots a_n) \in \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$? If yes, show it has the properties required of a subspace; if not, find an example that shows how a property of subspaces is violated:

(a) all α such that $a_1 \geq 0$;

(b) all α such that $a_1 + 3a_2 = a_3$;

(c) all α such that $a_2 = a_1^2$;

(d) all α such that $a_1 a_2 = 0$;

(e) all α such that α_2 is rational.

4 Let W be the set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfy

$$\begin{array}{rcrcrcrcrl} 2x_1 & - & x_2 & + & \frac{4}{3}x_3 & - & x_4 & & & = 0 \\ x_1 & & & & +\frac{2}{3}x_3 & & & - & x_5 & = 0 \\ 9x_1 & - & 3x_2 & + & 6x_3 & - & 3x_4 & - & 3x_5 & = 0. \end{array}$$

Find a finite set of vectors which spans W.

5 Let F be a field and let n be an integer with $n \geq 2$. Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A are subspaces of V? If yes, show it has the properties required of a subspace; if not, show how a property of subspaces is violated:

(a) all invertible A;

(b) all non-invertible A;

(c) all A such that $AB = BA$ where B is a fixed matrix in V;

(d) all A such that $A^2 = A$.

Section 2.3

2 Are the vectors $\alpha_1 = (1, 1, 2, 4)$, $\alpha_2 = (2, -1, -5, 2)$, $\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$ linearly independent in $\mathbb{R}^4$?

3 Find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors α_1, α_2, α_3 and α_4 in the preceding exercise.

4 Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, and $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors ϵ_1, ϵ_2 and ϵ_3 as a linear combination of α_1, α_2, and α_3.

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2.2 1. a) $S = \{(a_1, \dots a_n) : a_1 \geq 0\}$

is not a subspace, b/c

$(a_1, 0, \dots 0) \in S$ for $a_1 \geq 0$

but not

$-(a_1, 0, \dots 0) = (\underbrace{-a_1}_{<0}, 0, \dots 0)$.

b) $S = \{(a_1, \dots a_n) : a_1 + 3a_2 = a_3\}$

i) $(0, 0, \dots 0) \in S$ ✓

ii) if $\alpha, \beta \in S$ then for $c \in F$

$$a_1 + 3a_2 = a_3$$
$$cb_1 + 3cb_2 = cb_3$$

so

$$(a_1 + cb_1) + 3(a_2 + cb_2) = a_3 + cb_3$$

which means

$$(a_1 + cb_1, a_2 + cb_2, \dots) \in S$$

$$\alpha + c\beta \in S.$$

So S is a subspace

c) $S = \{(a_1, \dots a_n) : a_2 = a_1^2\}$

We have $(1, 1, 0, \dots 0) \in S$

but not $(c, c, 0, \dots 0)$

for $c \notin \{0, 1\}$, b/c then $c \neq c^2$,

so S is not a subspace.

d) $S = \{(a_1, a_2, \dots a_n) : a_1 a_2 = 0\}$

We have $(1, 0, 0, \dots 0) \in S$

and $(0, 1, 0, \dots) \in S$

but $(1, 1, 0, \dots) \notin S$

$= (1, 0, \dots 0)$

$+ (0, 1, \dots 0)$,

so S is not a subspace

e) $S = \{(a_1, a_2, \dots a_n) : a_2 \in \mathbb{Q}\}$

We have $(0, 1, 0, \dots 0) \in S$

but $(0, \pi, 0, \dots 0) \notin S$

$\parallel$

$\pi(0, 1, 0, \dots 0)$

so S is not a subspace.

4. Let $(x_1, x_2, \ldots x_5) \in \mathbb{R}^5$ satisfy

$$2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \tfrac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Row reduce

$$\begin{pmatrix} 2 & -1 & 4/3 & 0 & -1 \\ 1 & 0 & 2/3 & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2/3 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so after choosing x_3, x_4, x_5, we have determinded values of x_1, x_2

$$x_1 = -\tfrac{2}{3}x_3 + x_5$$

$$x_2 = -x_4 + 2x_5.$$

Spanning set is

$$\{(-\tfrac{2}{3}, 0, 1, 0, 0), (0, -1, 0, 1, 0),$$

$$(1, 2, 0, 0, 1)\}.$$

5. a) $S = \{ A \in F^{n \times n} : A \text{ invertible} \}$

is not a vector space b/c

O (zero-matrix) is not included

b) $S = \{ A \in F^{n \times n} : A \text{ non-invertible} \}$

Let

$$D_1 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{pmatrix} \Big\} n-1$$

$$D_2 = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & 1 \end{pmatrix} \Big\} n-1$$

then

D_1, D_2 are not invertible

but $D_1 + D_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is !

Not closed under addition.

c) $S = \{A \in F^{n \times n} : AB = BA\}$

for fixed B

i) 0 matrix is in S

ii) given $A_1, A_2 \in S$

we know $A_1 B = B A_1$,

$A_2 B = B A_2$,

so

$$(A_1 + cA_2) B$$
$$= A_1 B + c A_2 B$$
$$= B A_1 + c B A_2$$
$$= B(A_1 + cA_2)$$

which means $A_1 + c A_2 \in S$

$\Rightarrow$ S is a subspace!

d) $S = \{A \in F^{n \times n} : A^2 = A\}$

$$D = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in S$$

but

$$2D = \begin{pmatrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{pmatrix}$$ gives $(2D)^2 = 4D = 2(2D)$

So $2D \notin S$ $\Rightarrow$ not a subspace.

2.3 2. $\alpha_1 = (1,1,2,4)$, $\alpha_2 = (2,-1,-5,2)$

$\alpha_3 = (1,-1,-4,0)$, $\alpha_4 = (2,1,1,6)$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \begin{matrix} \\ \ominus \\ \\ \ominus \end{matrix} \sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & 2 & 6 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

does not have full rank, so vectors are linearly dependent.

3. Continue eliminating rows

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so basis vectors are

$$B = \{(1,1,2,4), (0,3,9,6)\}$$

4. Vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$, $\alpha_3 = (0,-3,2)$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & -3 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

has maximal rank $\Rightarrow$ $\{\alpha_1, \alpha_2, \alpha_3\}$ lin. indep.

Want to find $(P_{ij})_{i,j=1}^{3}$

so that $\varepsilon_j = \sum_{i=1}^{3} P_{ij}\, \alpha_i$.

We get

$$P = \begin{pmatrix} \frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

so

$$\varepsilon_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3$$

$$\varepsilon_2 = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3$$

$$\varepsilon_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 .$$