Department of Mathematics University of Houston

Math 4377

Advanced Linear Algebra

Fall 2008

Homework Set 6, due Thursday, Oct 2, 1pm

Section 2.3

6 Let V be the vector space of all 2×2 matrices over a field F. Prove that V has dimension 4 by finding a basis for V with 4 elements.

7 Let V be the vector space of the preceding exercise. Let W_1 be the set of matrices having the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

and the set W_2 containing all matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

10 Let V be a vector space over a field F. Assume that V is the span of finitely many vectors $\{\alpha_!, \alpha_2, \ldots \alpha_r\}$, $r \in \mathbb{N}$. Prove that V is finite-dimensional.

Section 2.4

1 Show that the vectors $\alpha_1 = (1,1,0,0)$, $\alpha_2 = (0,0,1,1)$, $\alpha_3 = (1,0,0,4)$, $\alpha_4 = (0,0,0,4)$ form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors ϵ_1, ϵ_2, ϵ_3 and ϵ_4 in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

2 Find the coordinate vector of $(1,0,1) \in \mathbb{C}^3$ relative to the ordered basis $\{(2i,1,0), (2,-1,1), (0,1+i,1-i)\}$.

7 Let V be the vector space of all polynomials from $\mathbb{R}$ to $\mathbb{R}$ of maximal degree 2, so each $f \in V$ is of the form

$$f(x) = c_0 + c_1 x + c_2 x^2,$$

with appropriate numbers c_0, c_1, $c_2 \in \mathbb{R}$.

For a given fixed number t, let

$$g_1(x) = 1, \quad g_2(x) = x + t \quad \text{and} \quad g_3(x) = (x+t)^2.$$

Prove that $B = \{g_1, g_2, g_3\}$ is a basis for V and compute the coordinates of the polynomial $f(x) = c_0 + c_1 x + c_2 x^2$ in this (ordered) basis B.

Math 4377

Homework Set 6, due

Oct 2, 2008

2.3

6. Let $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F \right\}$.

To show $\dim V = 4$, note we have a basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

b/c

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = x_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

has unique sol$^{\underline{n}}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

7. Let $W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} : x, y, z \in F \right\}$

$W_2 = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} : a, b, c \in F \right\}$.

a) W_1 is a subspace b/c

i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_1$

ii) if $\alpha, \beta \in W_1$, say

$$\alpha = \begin{pmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{pmatrix}$$

and $c \in F$,

then

$$c\alpha + \beta = \begin{pmatrix} cx_1 & -cx_1 \\ cy_1 & cz_1 \end{pmatrix} + \begin{pmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{pmatrix}$$

$$= \begin{pmatrix} cx_1 + x_2 & -(cx_1 + x_2) \\ cy_1 + y_2 & cz_1 + z_2 \end{pmatrix} \in W_1$$

W_2 is a subspace b/c

i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_2$

ii) if $\alpha, \beta \in W_2$, say

$$\alpha = \begin{pmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{pmatrix}$$

and $c \in F$

then

$$c\alpha + \beta = c\begin{pmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} ca_1 + a_2 & cb_1 + b_2 \\ -(ca_1 + a_2) & c_1 + c_2 \end{pmatrix} \in W_2 .$$

b) Find dimension by finding bases

$\dim W_1 = 3$ b/c

$$B_1 = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis, b/c

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = x_1 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

has a unique solution

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = \begin{pmatrix} x_1 & -x_1 \\ x_2 & x_3 \end{pmatrix} .$$

$\dim W_2 = 3$ b/c

$$B_2 = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis, similar to the basis for W_1.

$$\dim(W_1 \cap W_2) = \dim \left\{ \begin{pmatrix} x & -x \\ -x & z \end{pmatrix} : x, z \in \mathbb{R} \right\}$$

$$= 2$$

b/c $\left\{ \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is

a basis for $W_1 \cap W_2$, since

$$\begin{pmatrix} x & -x \\ -x & z \end{pmatrix} = x_1 \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

has unique solution $x_1 = x$, $x_2 = z$.

By dimension count formula,

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$= 3 + 3 - 2 = 4.$$

10. We know that if V is spanned by a set $\{\alpha_1, \dots \alpha_r\}$, then any linearly indep. set cannot have more than r vectors. Since each basis is linearly independent, it cannot have more than r vectors. Therefore, $\dim V \leq r$.

2.4

1. Row reduce

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow{\ominus} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{matrix} \\ \\ \times(-1) \\ \times\frac{1}{4} \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & +1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rank of matrix is 4, so vectors are linearly indep. Since we know the standard basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ has 4 vectors, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ must be a basis.

By inspection,

$$\begin{aligned} \varepsilon_1 &= \alpha_3 - \alpha_4 \\ \varepsilon_2 &= \alpha_1 - \alpha_3 - \alpha_4 \\ \varepsilon_3 &= \alpha_2 - \tfrac{1}{4}\alpha_4 \\ \varepsilon_4 &= \tfrac{1}{4}\alpha_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} [\varepsilon_1]_{\mathcal{B}} &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \\ [\varepsilon_2]_{\mathcal{B}} &= \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \\ [\varepsilon_3]_{\mathcal{B}} &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/4 \end{pmatrix} \\ [\varepsilon_4]_{\mathcal{B}} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/4 \end{pmatrix}. \end{aligned}$$

2. We want to solve

$$x_1 \begin{pmatrix} 2i \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1+i \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

augmented coeff. matrix

$$\begin{pmatrix} 2i & 2 & 0 & 1 \\ 1 & -1 & 1+i & 0 \\ 0 & 1 & 1-i & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -i & 0 & -i/2 \\ 0 & -1+i & 1+i & i/2 \\ 0 & 1 & 1-i & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -i & 0 & -i/2 \\ 0 & 1 & \frac{1+i}{-1+i} & \frac{i}{2(-1+i)} \\ 0 & 0 & \underbrace{1-i-\frac{1+i}{-1+i}}_{\underbrace{1-i+\frac{1}{\cancel{2}}(\cancel{2}i)}_{1}} & \underbrace{1-\frac{i}{2(-1+i)}}_{\frac{3}{4}+\frac{i}{4}} \end{pmatrix}$$

$$\Rightarrow \quad x_3 = \frac{3}{4} + \frac{i}{4}$$

$$\Rightarrow \quad x_2 = \underbrace{\frac{i}{2(-1+i)}}_{\frac{1}{4}-\frac{i}{4}} - \left(\frac{3}{4}+\frac{i}{4}\right)\underbrace{\left(\frac{1+i}{i-1}\right)}_{-i}$$

$$= -\frac{i}{4} + \frac{3}{4}i = \frac{i}{2}$$

$$\Rightarrow \quad x_1 = i\left(\frac{i}{2}\right) - \frac{i}{2} = -\frac{1}{2} - \frac{i}{2}$$

so
$$\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} -\frac{1}{2}-\frac{i}{2} \\ \frac{i}{2} \\ \frac{3}{4}+\frac{i}{4} \end{pmatrix}.$$

7. Given $f(x) = c_0 + c_1 x + c_2 x^2$,

we want to find d_0, d_1, d_2 s.th.

$$f(x) = d_0 + d_1(x+t) + d_2(x+t)^2.$$

Since highest power appears only with factor d_2, we know $c_2 = d_2$.

Equating all other powers

$$(1) \qquad c_0 = d_0 + d_1 t + d_2 t^2$$

$$c_0 = d_0 + d_1 t + c_2 t^2$$

$$(2) \qquad c_1 = d_1 + 2 d_2 t$$

$$= d_1 + 2 c_2 t$$

$$\text{so } (1') \qquad d_0 + d_1 t = c_0 - c_2 t^2$$

$$(2') \qquad d_1 = c_1 - 2c_2 t$$

$$\overset{\text{in } (1')}{\Rightarrow} (1'') \qquad d_0 = c_0 - c_2 t^2 - (c_1 - 2c_2 t)\,t$$

Thus we have shown any $f(x)$ can be expressed with a lin comb. of $\{1, x+t, (x+t)^2\}$. Since standard basis has 3 vectors $\{1, x, x^2\}$, we must have a basis.

Coordinates of $f(x)$ in new basis are

$$[f(x)]_B = \begin{pmatrix} c_2t^2 - c_1t + c_0 \\ -2c_2t + c_1 \\ c_2 \end{pmatrix}.$$