Department of Mathematics University of Houston

Math 4377

Advanced Linear Algebra

Fall 2008

Homework Set 7, due Tuesday, Oct 7, 1pm

Section 2.6

3 Consider the vectors $\alpha_1 = (-1, 0, 1, 2)$, $\alpha_2 = (3, 4, -2, 5)$, $\alpha_3 = (1, 4, 0, 9)$ in $\mathbb{R}^4$. Find a system of homogeneous linear equations for 4 unknowns x_1, x_2, x_3, x_4 such that the solutions give all vectors (x_1, x_2, x_3, x_4) that are in the subspace spanned by α_1, α_2 and α_3.

4 In $\mathbb{C}^3$, let $\alpha_1 = (1, 0, -i)$, $\alpha_2 = (1+i, 1-i, 1)$, and $\alpha_3 = (i, i, i)$. Prove that these three vectors form a basis $B = \{\alpha_1, \alpha_2, \alpha_3\}$ for $\mathbb{C}^3$. What are the coordinates of the vector $\gamma = (a, b, c)$ with respect to this (ordered) basis B?

x Consider another basis B' for $\mathbb{C}^3$ given by $\beta_1 = (1, i, i)$, $\beta_2 = (0, 1, i)$, and $\beta_3 = (0, 0, 1)$. (You do not have to prove it is a basis.) Find the change-of-coordinates matrix $P_B^{B'}$ from B to B' and compute the coordinate vector $[\gamma]_{B'}$ for the vector γ in the preceding problem.

Section 3.1

1 Which of the following functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ are linear transformations? Explain why/why not.

(a) $T(x_1, x_2) = (1 + x_1, x_2)$,

(b) $T(x_1, x_2) = (x_2, x_1)$,

(c) $T(x_1, x_2) = (x_1^2, x_2)$,

(d) $T(x_1, x_2) = (\sin x_1, x_2)$,

(e) $T(x_1, x_2) = (x_1 - x_2, 0)$.

3 Let V be the vector space of all polynomials over $\mathbb{R}$, meaning all functions of the form $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, $n \in \mathbb{N}$. Consider the linear map $T : V \to V$ which maps p to $p'(x) = c_1 + 2c_2 x + \cdots + nc_n x^{n-1}$ (differentiation). Describe the range and the null space for this differentiation transformation. Do the same for the integration map $S : V \to V$ which maps p to $P(x) = \int_0^x p(s) ds$.

4 Is there a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$? If so, find such a T by giving $T(a, b, c)$. If not, explain why it does not exist.

Math 4377

Homework Set 7,

due Oct 7, 2008

2.6

3. We want to find all $\xi = (x_1, x_2, x_3, x_4)$ such that there are $\{c_1, c_2, c_3\}$ and

$$c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3 = \xi .$$

Note: The span of row vectors does not change under row operations, so

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 4 & 1 & 11 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

means $\beta_1 = (-1, 0, 1, 2)$ and
$\beta_2 = (0, 4, 1, 11)$

have same span, so

$$c_1' \beta_1 + c_2' \beta_2 = \xi ,$$

$$\Rightarrow \quad (-c_1', 4c_2', c_1' + c_2', 2c_1' + 11c_2') = \xi$$

$$= (x_1, x_2, x_3, x_4)$$

We conclude x_1, x_2 are free variables and

$$x_3 = -x_1 + \frac{1}{4} x_2$$

$$x_4 = -2x_1 + \frac{11}{4} x_2 .$$

4. Given $\alpha_1 = (1, 0, -i)$, $\alpha_2 = (1+i, 1-i, 1)$

$\alpha_3 = (i, i, i)$.

To show $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis, verify that linear system $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ has unique solution, the coordinates.

$$\left(\begin{array}{ccc|c} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ -i & 1 & i & c \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ 0 & \cancel{1}+i-\cancel{1} & i-1 & ia+c \end{array}\right) \begin{array}{l} \\ \times \frac{1}{2}(1+i) \\ \oplus \times(-i) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1+i & i & a \\ 0 & 1 & \frac{1}{2}(i-1) & \frac{b}{2}(1+i) \\ 0 & 0 & \frac{3i}{2} - \frac{1}{2} & ia + c + \frac{1}{2} b (1-i) \end{array}\right)$$

$$\sim \cdots \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \left(\frac{1}{5} - \frac{2i}{5}\right)(a + b - 2c) \\ 0 & 1 & 0 & \left(\frac{1}{5} - \frac{2i}{5}\right)(a - (1-i)b - ic) \\ 0 & 0 & 1 & \left(\frac{1}{5} - \frac{2i}{5}\right)((1+i)a - ib + (1-i)c) \end{array}\right)$$

so $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis, and coordinates of (a,b,c) are

$$[\gamma]_{\mathcal{B}} = \begin{pmatrix} \left(\frac{1}{5} - \frac{2i}{5}\right)(a+b-2c) \\ \left(\frac{1}{5} - \frac{2i}{5}\right)(a-(1-i)-ic \\ \left(\frac{1}{5} - \frac{2i}{5}\right)((1+i)a - ib + (1-i)c \end{pmatrix}.$$

x. To find $P_{\mathcal{B}}^{\mathcal{B}'}$, consider

$\alpha_1, \alpha_2, \alpha_3$ and compute coordinates in basis $\mathcal{B}'$.

- $$\begin{pmatrix} i \\ i \\ i \end{pmatrix} = i \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} + (1+i) \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  must be all other vectors have zero

  gets 2nd entry right

- $$\begin{pmatrix} 1+i \\ 1-i \\ 1 \end{pmatrix} = (1+i) \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} + 2(1-i) \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} - 3i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} = 1 \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} + (-i) \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} + (-1-2i) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

these coefficients are the colums of $P_{\mathcal{B}}^{\mathcal{B}'}$

$$P_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 1 & 1+i & i \\ -i & 2(1-i) & 1+i \\ -1-2i & -3i & 2 \end{pmatrix}$$

Now, change of coordinates gives

$$[\gamma]_{\mathcal{B}'} = P_{\mathcal{B}}^{\mathcal{B}'} [\gamma]_{\mathcal{B}}$$

$$= \begin{pmatrix} 1 & 1+i & i \\ -i & 2(1-i) & 1+i \\ -1-2i & -3i & 2 \end{pmatrix} \begin{pmatrix} (\frac{1}{5} - \frac{2i}{5})(a+b-2c) \\ (\frac{1}{5} - \frac{2i}{5})(a-(1-i)b-ic) \\ (\frac{1}{5} - \frac{2i}{5})((1+i)a-ib+(1-i)c) \end{pmatrix}$$

$$= \begin{pmatrix} a \\ -ia+b \\ -(1+i)a-ib+c \end{pmatrix}.$$

3.1

1. a) $T(0,0) = (1,0) \neq (0,0)$ not linear

b) $T(x_1, x_2) = (x_2, x_1)$ linear b/c

$T(0,0) = (0,0)$ ✓

$T((x_1,x_2) + c(y_1,y_2))$

$= (x_2 + cy_2, x_1 + cy_1)$

$= (x_2, x_1) + c(y_2, y_1)$

$= T(x_1,x_2) + cT(y_1,y_2)$ ✓

c) $T(0,0) = (0,0)$ ✓

$T(2,0) = (4,0) \neq 2T(1,0) = (2,0)$

not linear

(d) $T(\frac{\pi}{2}, 0) = (1,0) \neq \frac{1}{2} T(\pi, 0)$

$= (0,0)$

not linear

(e) $T(x_1, x_2) = (x_1 - x_2, 0)$

$T(0,0) = (0,0)$ ✓

$T((x_1, x_2) + c(y_1, y_2))$

$= T(x_1 + cy_1, \; x_2 + cy_2)$

$= (x_1 + cy_1 - x_2 - cy_2, 0)$

$= (x_1 - x_2, 0) + c(y_1 - y_2, 0)$

$= T(x_1, x_2) + c\,T(y_1, y_2)$ linear.

3. $T: p \mapsto p'$

$\ker(T) = \{ p : p'(x) = 0 \}$

$p \in \ker(T) \Rightarrow p(x) = c_0$ (all constant functions)

$p \in \operatorname{ran}(T) \Rightarrow p(x) = c_1 + 2c_2 x + \dots + n c_n x^{n-1}$

$= c_0' + c_1' x + c_2' x^2 + \dots + c_{n-1}' x^{n-1}$

(so all polynomials are in range)

$S: \quad p \mapsto P, \qquad P(x) = \int_0^x p(s)\,ds$

$$\Rightarrow \quad P(x) = c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \dots + \frac{c_n}{n+1} x^{n+1}$$

$\ker(S) = \{ p : P(x) = 0 \}$

$\Rightarrow$ $p \in \ker(S)$ means $c_0 = c_1 = \dots = c_n = 0$

$\Rightarrow$ $p(x) = 0$ (only zero polynomial)

$\operatorname{ran}(S) = \{ P : P(x) = c_0 x + \frac{c_1}{2} x^2 + \dots + \frac{c_n}{n+1} x^{n+1} \}$

$= \{ P : P(x) = c_1' x + c_2' x^2 + \dots + c_{n+1}' x^{n+1} \}$

$P \in \operatorname{ran}(S)$ $\Rightarrow$ P has no constant term,

i.e. $P(0) = 0$.

4. Want $T(1,-1,1) = (1,0)$,

$T(1,1,1) = (0,1)$.

Note $(1,-1,1)$ and $(1,1,1)$ are linearly independent, so extend to basis, say

$\mathcal{B} = \{ (1,-1,1), (1,1,1), (0,0,1) \}$

Now prescribing images of vectors in $\mathcal{B}$ defines a linear transf$^{\underline{n}}$ T, choose

e.g.

$$T(1,-1,1) = (1,0)$$
$$T(1,1,1) = (0,1)$$
$$T(0,0,1) = (0,0).$$

Now we want to find map $\mathbb{R}^3 \to \mathbb{R}^2$ given by $T(\vec{x}) = A\vec{x}$.

$$\underbrace{A}_{2\times 3 \text{ matrix}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(1) $\quad A_{11} - A_{12} + \cancel{A_{13}} = 1$

(2) $\quad A_{21} - A_{22} + \cancel{A_{23}} = 0$

(3) $\quad A_{11} + A_{12} + \cancel{A_{13}} = 0$

(4) $\quad A_{21} + A_{22} + \cancel{A_{23}} = 1$

$$A_{13} = 0$$
$$A_{23} = 0$$

$\overset{(1)+(3)}{\Rightarrow} \quad 2A_{11} = 1 \quad \Rightarrow \quad A_{11} = \frac{1}{2}$

$\overset{-(1)+(3)}{\Rightarrow} \quad 2A_{12} = -1 \quad \Rightarrow \quad A_{12} = -\frac{1}{2}$

$$\overset{(2)+(4)}{\Rightarrow} \quad 2A_{21} = 1 \quad \Rightarrow \quad A_{21} = \frac{1}{2}$$

$$\overset{-(+2)+(4)}{\Rightarrow} \quad 2A_{22} = 1 \quad \Rightarrow \quad A_{22} = \frac{1}{2}$$

So matrix A is

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$