Math 4377

Homework Set 8

due Tue, 10/28

3.3 2 a) We note ~~if~~ there are $c_2, c_3 \in \mathbb{C}$ s.th.

$$\alpha_1 = c_2\alpha_2 + c_3\alpha_3$$

if and only if

$$T(\alpha_1) = c_2 T(\alpha_2) + c_3 T(\alpha_3)$$

for the same c_2, c_3.

Therefore, consider system

$$\left(\begin{array}{cc|c} -2 & -1 & 1 \\ 1+i & 1 & 0 \\ 0 & 1 & i \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} -2 & -1 & 1 \\ 0 & \frac{1}{2}-\frac{i}{2} & \frac{1}{2}+\frac{i}{2} \\ 0 & 1 & i \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} -2 & -1 & 1 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} c_3 = i \\ c_2 = -\frac{1}{2} - \frac{i}{2} \end{array}$$

so $\alpha_1 \in \operatorname{span}\{\alpha_2, \alpha_3\}$

b) We now know

$$i\alpha_3 = \alpha_1 - \left(-\tfrac{1}{2} - \tfrac{i}{2}\right)\alpha_2$$

so $\alpha_3 \in \operatorname{span}\{\alpha_1, \alpha_2\}$

Repeating the above for testing whether $\alpha_4 \in \operatorname{span}\{\alpha_1, \alpha_2\}$, which is equivalent to $T(\alpha_4) \in \operatorname{span}\{T(\alpha_1), T(\alpha_2)\}$, we get

$$\left(\begin{array}{cc|c} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & 2i & 3-\sqrt{2}i \end{array}\right) \qquad \begin{aligned}(1+i)^2 &= 1+i+i+i^2 \\ &= 2i\end{aligned}$$

$$\sim \left(\begin{array}{cc|c} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & \cancel{1+i}\,0 & 4-(1+\sqrt{2})i \end{array}\right)$$

so no such coeffs. exist.

This means α_3 spans the intersection,

$$\mathbb{C}\alpha_3 = W_1 \cap W_2 .$$

Alternative: Solve for $c_1, c_2, \dots, c_4$ to get

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) = c_3 T(\alpha_3) + c_4 T(\alpha_4) \iff c_1\alpha_1 + c_2\alpha_2 = c_3\alpha_3 + c_4\alpha_4$$

c) By our preceding computations,

$$\alpha_4 \notin \text{span}\{\alpha_1, \alpha_2\}, \quad \alpha_3 \in \text{span}\{\alpha_1, \alpha_2\}$$

so, since $\{\alpha_1, \alpha_2\}$ are linearly indep. (no scalar multiples of each other), we know

$$\{\alpha_1, \alpha_2, \alpha_4\}$$

is a basis.

3. We claim that

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}$$

is a basis for W.

It is clear that every $A \in W$ is a linear comb. of elements in B.

Now, test lin. independence: If $\exists\, a, b, c, d \in \mathbb{R}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$+ c \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} -a+d & b+ic \\ b-ic & a+d \end{pmatrix}$$

$$\Rightarrow \quad a+d=0, \quad a-d=0 \quad \Rightarrow \quad a=d=0$$

(no imaginary parts on diagonal, b/c
$A_{i,i} = \overline{A_{i,i}}$)

also $b+ic = 0 \Rightarrow b = c = 0$.

We verify

$$T(1,0,0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T(0,1,0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T(0,0,1,0) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$T(0,0,0,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so T maps the standard basis to a basis of W, which means it is an isomorphism.

3.4 2a) By

$$T(1,0,0) = (1,-1)$$
$$T(0,1,0) = (1,0)$$
$$T(0,0,1) = (0,2)$$

we have

$$[T]_{B}^{B'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

b) Compute change of basis matrix from B to E (std. basis)

$$P_{B}^{E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$P_{E'}^{B'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow [T]_{B}^{B'} = [id]_{E'}^{B'} [T]_{E}^{E'} [id]_{B}^{E}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}.$$

x) Let $A = [T]_{\mathcal{B}}^{\mathcal{B}}$.

Method A Find ran(A) first:

$$A\vec{x} = x_1 A\vec{e}_1 + x_2 A\vec{e}_2 + x_3 A\vec{e}_3$$

$$= \underset{\uparrow}{x_1 A_1} + \underset{\uparrow}{x_2 A_2} + \underset{\uparrow}{x_3 A_3}$$

columns of A

Take ~~row~~ matrix w/ row vectors

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Span of ran(A) (!) has basis

$$\{ (1,0,-1), (0,1,5) \}$$

Basis for ran(T) is $\{\beta_1 - \beta_3, \beta_2 + 5\beta_3\} = \{(1,1,0), (0,1,6)\}$

For null space, want

$$A\vec{x} = \vec{0} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

free

$\Rightarrow \quad x_2 = -x_3, \quad x_1 = -2x_2 - x_3$

$$= x_3$$

$\Rightarrow \quad \ker(T) = \operatorname{span}\{\beta_1 - \beta_2 + \beta_3\}$

so $\ker(T)$ has basis

$$\{(1, 0, 1)\}.$$

x) We have $T(\vec{x}) = A'\vec{x}$

Method $\underline{B}$ with

$$A' = [id]_{\mathcal{B}}^{E} A \; [id]_{E}^{\mathcal{B}}.$$

We have

$$[id]_{\mathcal{B}}^{E} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$[id]_{E}^{\mathcal{B}} = \left([id]_{\mathcal{B}}^{E}\right)^{-1}$$

Find it

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\Rightarrow \quad A' = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ -2 & 1 & 2 \\ -6 & 0 & 6 \end{pmatrix}$$

Now find basis for range

take matrix with row vectors

$$\begin{pmatrix} -1 & -2 & -6 \\ 1 & 1 & 0 \\ 1 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & -6 \\ 0 & -1 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

so basis for range of T is

$$\{ (1,2,6), \ (0,1,6) \} .$$

To find kernel,

$$\begin{pmatrix} -1 & 1 & 1 \\ -2 & 1 & 2 \\ -6 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -6 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 = x_3 \\ x_2 = 0 \\ \Uparrow \end{array}$$

$\uparrow$ free variable x_3

basis for kernel is given by

$$\{ (1,0,1) \} .$$