Math 4377

Homework Set 9

3.5 1.a) Let $f(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3$.

We want $f(\alpha_1) = 1$, $f(\alpha_2) = -1$, $f(\alpha_3) = 3$.

System for coeffs c_1, c_2, c_3 is

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ -1 & -1 & 0 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & -1 & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$\Rightarrow$ $c_1 = 4$, $c_2 = -7$, $c_3 = -3$

$\Rightarrow$ $f(a, b, c) = 4a - 7b - 3c$.

b) Repeat with, say $f(\alpha_3) = 1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \quad c_2 = -1 \quad \Rightarrow 2c_3 = c_2 \quad \Rightarrow \quad c_3 = -\tfrac{1}{2}$$

$$c_1 = -c_3 = \tfrac{1}{2}$$

so, e.g.

$$f(a,b,c) = \tfrac{1}{2}a - b - \tfrac{1}{2}c$$

has the desired properties.

2. We want $f_i(\alpha_j) = \delta_{ij}$.
Writing $f_i(x_1, x_2, x_3) = c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3$ and forming matrix C with i-th row $(c_{i1} \; c_{i2} \; c_{i3})$, as well as matrix

$$A = (\alpha_1 \; \alpha_2 \; \alpha_3)$$

↑ ↑ ↑
columns

, we want

$$CA = I.$$

So $C = A^{-1}$!

Compute inverse

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -\frac{1}{2} \end{array}\right).$$

The dual basis $\{f_1, f_2, f_3\}$ is given by

$$f_1(x_1, x_2, x_3) = x_1 + x_2 - \tfrac{1}{2}x_3$$

$$f_2(x_1, x_2, x_3) = -x_1 - x_2 + x_3$$

$$f_3(x_1, x_2, x_3) = x_2 - \tfrac{1}{2}x_3 .$$

x Let $V = \mathcal{P}_2(\mathbb{R})$ and consider

$$f_n = \int_{-1}^{1} x^{n-1} p(x)\, dx \quad \text{for } p \in \mathcal{P}_2(\mathbb{R}).$$

Construct basis for which $\{f_1, f_2, f_3\}$ is dual.

If $p(x) = c_0 + c_1 x + c_2 x^2$

we have

$$f_1(p) = 2c_0 + c_2 \underbrace{\int_{-1}^{1} x^2\, dx}_{\frac{2}{3}}$$

$$f_2(p) = \frac{2}{3} c_1$$

$$f_3(p) = \frac{2}{3} c_0 + \frac{2}{5} c_2 .$$

So for standard basis $E = \{1, x, x^2\}$,

~~and~~

$$\begin{pmatrix} f_1(p) \\ f_2(p) \\ f_3(p) \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}}_{A} [p]_E$$

To have ~~of~~ $\{p_1, p_2, p_3\}$ s.th. $f_j(p_k) = \delta_{jk}$, we compute A^{-1}

$$\left(\begin{array}{ccc|ccc} 2 & 0 & \frac{2}{3} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 0 & \frac{2}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{8}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \frac{8}{15} & 0 & 0 & \frac{9}{15} & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{15}{8} & 0 & \frac{45}{8} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{8} & 0 & -\frac{15}{8} \\ 0 & 1 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & -\frac{15}{8} & 0 & \frac{45}{8} \end{array}\right)$$

so

$$[p_1]_E = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9/8 \\ 0 \\ -15/8 \end{pmatrix}$$

$$[p_2]_E = \begin{pmatrix} 0 \\ 3/2 \\ 0 \end{pmatrix}, \quad [p_3]_E = \begin{pmatrix} -15/8 \\ 0 \\ 45/8 \end{pmatrix}$$

Polynomials are

$$p_1(x) = \frac{9}{8} - \frac{15}{8} x^2$$

$$p_2(x) = \frac{3}{2} x$$

$$p_3(x) = -\frac{15}{8} + \frac{45}{8} x^2 .$$

7 For any $x = d_1 \alpha_1 + d_2 \alpha_2$, $\quad d_1, d_2 \in \mathbb{R}$

we have

$$f(x) = f(d_1 \alpha_1 + d_2 \alpha_2)$$

$$= d_1 f(\alpha_1) + d_2 f(\alpha_2)$$

so $f \in W^\circ \iff f(\alpha_1) = f(\alpha_2) = 0$.

To obtain c_j's we solve

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$\uparrow \quad \uparrow$

free variables

so $c_3, c_4 \in \mathbb{R}$ are arbitrary

and $c_1 = c_3 - 2c_4$

$c_2 = -c_3 - c_4$.

11 Let W_1, W_2 be subspaces of a finite-dim V

a) Prove $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$

Given f such that

$$f(x+y) = 0 \quad \text{for all} \quad x \in W_1,\ y \in W_2$$

then setting $x = 0$ or $y = 0$ gives

$$f(x) = 0 \quad \text{for all} \quad x \in W_1$$

and $$f(y) = 0 \quad \text{for all} \quad y \in W_2$$

$$\Rightarrow \quad f \in W_1^0 \cap W_2^0.$$

Conversely assuming this is the case,

then $$f(x+y) = \underbrace{f(x)}_{0} + \underbrace{f(y)}_{0} = 0$$

so $f \in W_1^0 \cap W_2^0 \Rightarrow f \in (W_1 + W_2)^0$.

b) Prove $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

Method A:

we prove

Equivalently, $(W_1 \cap W_2)^{00} \overset{?}{=} (W_1^0 + W_2^0)^0$

$$\overset{a)}{=} W_1^{00} \cap W_2^{00}$$

but $W_1, W_2, W_1 \cap W_2$ are subspaces,

so result follows from $((W_1 \cap W_2))^{00} = W_1 \cap W_2$

and $W_1^{00} = W_1$, $W_2^{00} = W_2$.

Method B:

We first show $(W_1 \cap W_2)^0 \supset W_1^0 + W_2^0$.

Given $f \in W_1^0 + W_2^0$, then

$$f = g + h \quad \text{with} \quad g \in W_1^0,\ h \in W_2^0.$$

Now let $\alpha \in W_1 \cap W_2$, then

$$f(\alpha) = \underbrace{g(\alpha)}_{\substack{0 \\ \text{b/c } \alpha \in W_1^0}} + \underbrace{h(\alpha)}_{\substack{0 \\ \text{b/c } \alpha \in W_2^0}} = 0$$

so $f \in (W_1 \cap W_2)^0$.

Now, by dimension counting

$$\dim (W_1 \cap W_2)^0 = \dim V - \dim (W_1 \cap W_2)$$

and

$$\begin{aligned}
\dim (W_1^0 + W_2^0) &= \dim W_1^0 + \dim W_2^0 \\
&\quad - \dim \big(\underbrace{W_1^0 \cap W_2^0}_{\text{by e) } (W_1 + W_2)^0}\big) \\
&= \dim V - \dim W_1 \\
&\quad + \cancel{\dim V} - \dim W_2 \\
&\quad - \cancel{\dim V} + \dim (W_1 + W_2) \\
&= \dim V - \dim (W_1 \cap W_2).
\end{aligned}$$

Since $W_1^0 + W_2^0 \subset (W_1 \cap W_2)^0$

and both have the same dimension,

we know

$$W_1^0 + W_2^0 = (W_1 \cap W_2)^0 .$$