Math 4397/6397, Fall 2009
Problem Set 1, due Thursday, Sep 3

Solutions

Problem 1 a. We have $1 = P(\Omega) = P(\Omega \cup \emptyset)$. Also notice that Ω and $\emptyset$ are mutually exclusive (their intersection is empty). By additivity, $1 = P(\Omega) + P(\emptyset) = 1 + P(\emptyset)$. Solving for $P(\emptyset)$ provides the result.

b. From $B = (B \cap A) \cup (B \cap A^c)$, we get $P(B) = P((B \cap A) \cup (B \cap A^c)) = P(B \cap A) + P(B \cap A^c)$. Now since $A \subset B$ we have that $B \cap A = A$ and hence $P(B) = P(A) + P(B \cap A^c)$ hence $P(B) \geq P(A)$.

c. We know $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$ and these sets are mutually exclusive. Using the "double counting" argument as in class gives $P(A) + P(B) = P(A \cup B) + P(A \cap B)$, which gives the result.

d. Again, we use $A = (A \cap B) \cup (A \cap B^c)$, and that the sets in parentheses are disjoint (mutually exclusive).

e. Use induction, or prove directly. Each E_j is a subset of $\cup_{i=1}^n E_i$ for any j. Therefore using Part b, we know that $P(E_j) \leq P(\cup_{i=1}^n E_i)$ for all j. Since the maximum of $\{P(E_i)\}$ is achieved for some i, this produces the result.

Problem 2 Brief solutions (fill in the blanks yourselves)

a. Let A and B be the events that the father and mother contract influenza respectively. We know $P(A \cup B) = .17$, $P(B) = .12$, and $P(A \cap B) = .06$. Now use the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to solve for $P(B)$.

b. Use the fact that $P(A^c \cap B^c) = 1 - P(A \cup B)$.

c. Use the fact that $P(A^c \cap B) + P(A \cap B) = P(B)$.

d. Use the fact that $P(A \cap B^c) + P(A \cap B) = P(A)$.

Problem 3 a. We know $e^{-x} > 0$ for all x, so $f(x) = e^{-x}/(1 + e^{-x})^2 > 0$ as well. Also,

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{(1 + e^{-x})^2} dx = \left. \frac{1}{1 + e^{-x}} \right|_{-\infty}^{\infty} = 1 - 0 = 1.$$

b. Integrating again gives

$$F(x) = \int_{-\infty}^{x} \frac{e^{-t}}{(1 + e^{-t})^2} dt = \left. \frac{1}{1 + e^{-t}} \right|_{-\infty}^{x} = \frac{1}{1 + e^{-x}}.$$

c. For $x = 0$, we have $F(0) = 1/2$. Thus, the probability of $X \leq 0$ is $1/2$. (In other words, 0 is the median.)

d. Starting from

$$p = F(x_p) = \frac{1}{1 + e^{-x_p}}$$

and solving for x_p gives

$$\begin{aligned}
p(1+e^{-x_p}) &= 1 \\
p + pe^{-x_p} &= 1 \\
pe^{-x_p} &= 1-p \\
e^{-x_p} &= \frac{1-p}{p} \\
x_p &= -\log((1-p)/p) = \log(p/(1-p))
\end{aligned}$$