

Math 4397/6397, Fall 2009
Problem Set 2, due Thursday, Sep 10

Solutions

Problem 1. We compute

$$\begin{aligned}\text{Var}(x) &= E[(X - \mu)^2] \\ &= E[X^2 - 2X\mu + \mu^2] \\ &= E[X^2] - 2\mu E[X] + E[\mu^2] \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2.\end{aligned}$$

Problem 2. Given

$$f(x) = \begin{cases} cx^k & , 0 \leq x \leq 1 \\ 0 & , \text{else} \end{cases}$$

for some constant $k \geq 0$.

a. We know $c \geq 0$ because $f(x) \geq 0$. To determine c we integrate

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^1 cx^k dx = \frac{c}{k+1} x^{k+1} \Big|_0^1 = \frac{c}{k+1}.$$

For a probability density this should be one and we must therefore have $c = k+1$.

b. Integrating as above gives $F(x) = x^{k+1} + C$ for $x \in [0, 1]$ with a C we have to determine. Since f vanishes outside of this interval, F is constant on $x \geq 1$ and $x \leq 0$. This implies $F(1) = F(\infty) = 1 + C$, which gives $C = 0$.

c. $F(x_p) = (x_p)^{k+1} = p$ gives $x_p = p^{1/(k+1)}$.

d. Using the CDF, we obtain $P(a < X < b) = F(b) - F(a) = b^{k+1} - a^{k+1}$.

e. The mean is

$$\mu = E[X] = \int_0^1 x(k+1)x^k dx = \frac{k+1}{k+2} x^{k+2} \Big|_0^1 = \frac{k+1}{k+2}$$

and the variance

$$E[(X-\mu)^2] = E[X^2] - \mu^2 = \int_0^1 x^2(k+1)x^k dx - \mu^2 = \frac{k+1}{k+3} x^{k+3} \Big|_0^1 - \mu^2 = \frac{k+1}{k+3} - \frac{(k+1)^2}{(k+2)^2}.$$

This simplifies to

$$E[(X - \mu)^2] = \frac{k+1}{(k+2)^2(k+3)}.$$

Problem 3. Since s and $1 - s$ are both non-negative, $sf_1(x) + (1 - s)f_2(x) \geq 0$ for all $x \in \mathbb{R}$. By the linearity of the integral, we have $\int_{-\infty}^{\infty} (sf_1(x) + (1 - s)f_2(x))dx = s \int_{-\infty}^{\infty} f_1(x)dx + (1 - s) \int_{-\infty}^{\infty} f_2(x)dx = s + (1 - s) = 1$. The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\{sf_1(x) + (1 - s)f_2(x)\}dx \\ &= s \int_{-\infty}^{\infty} xf_1(x) + (1 - s) \int_{-\infty}^{\infty} xf_2(x)dx \\ &= s\mu_1 + (1 - s)\mu_2 \end{aligned}$$

To solve the variance we first calculate $E[X^2]$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2\{sf_1(x) + (1 - s)f_2(x)\}dx \\ &= s \int_{-\infty}^{\infty} x^2 f_1(x) + (1 - s) \int_{-\infty}^{\infty} x^2 f_2(x)dx \\ &= s(\sigma_1^2 + \mu_1^2) + (1 - s)(\sigma_2^2 + \mu_2^2) \end{aligned}$$

Therefore the variance is

$$E[X^2] - (E[x])^2 = s(\sigma_1^2 + \mu_1^2) + (1 - s)(\sigma_2^2 + \mu_2^2) - \{s\mu_1 + (1 - s)\mu_2\}^2.$$

Problem 4. a. Denote X_i the earnings in the i th round, then the total earnings are $\sum_{i=1}^{10} X_i$. For each round, assuming the coin is unbiased, we have $P(X_i = 1) = P(X_i = -1) = 1/2$, and thus we get the expected value $E[X_i] = (1)(1/2) + (-1)(1/2) = 0$. Using linearity of expected values,

$$E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 0.$$

b. The variance for one coin flip is $\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = 1 - 0$, so for 10 flips, assuming independence gives $\text{Var}(\sum_{i=1}^{10} X_i) = \sum_{i=1}^{10} \text{Var}(X_i) = 10$.

c. Now we have $E[X_i] = (-1)(0.6) + (1)(0.4) = -0.2$, so $E[\sum_{i=1}^{10} X_i] = 10E[X_1] = -2$. The variance for one flip is $\text{Var}(X_i) = E[X_i^2] - \mu^2 = 1 - 0.04 = 0.96$, so again by independence $\text{Var}(\sum_{i=1}^{10} X_i) = 10\text{Var}(X_1) = 9.6$.

Problem 5. The easiest way to do this calculation is to calculate the probability that a person remains infection free after n swims. Assuming that infections occur independently for each swim we have

$$P(\text{Infection free after } n \text{ swims}) = \left(\frac{499}{500}\right)^n.$$

An hence the probability of getting infected is

$$1 - (499/500)^n.$$

Setting this equal to .5 and solving for n yields $n = \log(.5)/\log(499/500)$ which is 346.23. Therefore 346 acts yields a probability just under .5 and 347 yields a probability just over .5 so the answer is 347.