

Information Theory with Applications

MATH 6397 – Fall 2008

November 11, 2008

Homework Set 3, due Tu Nov 18, 2008

1. Optimal quantization. Let X be a random variable which is normally distributed with mean zero and variance σ^2 . Which map $\phi : \mathbb{R} \rightarrow \{a, b\}$ and choice of the two numbers $a < b$ minimizes the mean-square error $\mathbb{E}[(X - \phi(X))^2]$?
2. Entropy maximizers. Prove that among all non-negative random variables with a given mean $\mu > 0$, the exponential distribution has maximal differential entropy.
3. Rate distortion theory for binary random variables.

- (a) Let X be a random variable with values in $\mathbb{A} = \{0, 1\}$ and $\mathbb{P}(X = 0) = p < 1/2$. Assume $0 \leq \epsilon \leq p$. Show that for any random variable Y with the same alphabet and $\mathbb{P}(X \neq Y) \leq \epsilon$, we have

$$I(X; Y) \geq h(p) - h(\epsilon)$$

where h is the binary entropy.

- (b) Let X be as before, and assume $0 \leq \epsilon \leq p < 1/2$. Define a random variable Y such that it has the marginal distribution satisfying

$$\mathbb{P}(Y = 0) = \frac{1 - p - \epsilon}{1 - 2\epsilon},$$

and conditioning on the outcome of Y gives $\mathbb{P}(X = 0|Y = 0) = 1 - \epsilon$, $\mathbb{P}(X = 0|Y = 1) = \epsilon$. Prove that

$$I(X; Y) = h(p) - h(\epsilon).$$

- (c) Given a discrete memoryless source $\{X_j\}_{j=1}^{\infty}$ with alphabet $\mathbb{A} = \{0, 1\}$ and $\mathbb{P}(X_1 = 0) = p < 1/2$, use the preceding parts to show that for the additive Hamming distortion (d_n counts the number of errors for binary input/output sequences of length n),

$$R(D) = \begin{cases} h(p) - h(D), & \text{if } 0 \leq D < p \\ 0, & \text{if } D > p \end{cases}.$$

4. Vector channel

Consider 3 parallel additive noise channels which can receive a total expected input power of at most S . The input and output vectors (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are related by $Y_i = X_i + N_i$, where the N_i 's are Gaussian, independent of the input, have mean zero and the covariance matrix for (N_1, N_2, N_3) is

$$\text{Cov}_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}$$

with $-1 \leq \rho \leq 1$. Find the capacity for this channel. Show that it is enough to consider $\rho > 0$. Distinguish cases depending on whether S is smaller/greater than ρ or 3ρ .