Stochastic Processes - Spring 2008

Practice Problems for Final Exam

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Duration: 150 minutes

First Name: ______________

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Show all work. No points will be given for numerical answers without working being shown.

(1) Consider the (continuous-time) Poisson process $\{N_t\}_{t\geq 0}$, which has independent increments on disjoint intervals, with distribution given by

$$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}$$

for all $s, t \geq 0, k \in \{0, 1, 2, \ldots\}$ and a fixed parameter $\lambda > 0$. Show that the process $M_t = N_t - \lambda t$ is a martingale.

(2) Suppose $\{B_t\}_{t\geq 0}$ is a standard Brownian motion starting at $B_0 = 0$. Let $\widetilde{B}_t = tB_1 - B_t$.

(a) Compute $\mathbb{E}[\widetilde{B}_s B_t]$. Give a reason why B_1 is independent of $\sigma(\{\widetilde{B}_s : 0 \leq s \leq 1\})$.

(b) Compare for fixed $s \in [0, 1]$ the distributions of $\widetilde{B}_s$ and $\widetilde{B}_{1-s}$.

(3) Let W_t be a Brownian motion with drift parameter μ, that is $W_t = B_t + \mu t$.

(a) Show that for any real $\lambda > 0$

$$V(t) = e^{\lambda W_t - (\lambda\mu + \frac{\lambda^2}{2})t}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(\{W_s : 0 \le s \le t\})$.

(b) Taking $\lambda = -2\mu$ in (a) you may conclude that $V_0(t) = e^{-2\mu W(t)}$ is a martingale. By using a stopping time argument or otherwise show that the probability that the Brownian motion with drift μ reaches $b > 0$ before $a < 0$ is

$$\frac{1 - e^{-2\mu a}}{e^{-2\mu b} - e^{-2\mu a}}\,.$$

(4) (a) Let B_t denote standard Brownian motion and $\mathcal{F}_t$ the σ-algebra generated by the random variables $\{B_s\}_{0 \le s \le t}$.

Let $Y_t = \max_{0 \le s \le t} B_s$. Use the reflection principle to show that for all $t \ge 0$, the distribution of Y_t is identical to that of $|B_t|$.

(b) Compute for fixed $t \ge 0$, the family of random variables $\{S_h\}_{0<h<1}$

$$S_h = \frac{B_{t+h} - B_t}{h}\,.$$

Show that this family has diverging norm in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$\sup_{0<s<1} \mathbb{E}[S_h^2] = \infty\,.$$

(5) Suppose that $\{X_n\}$ is a Markov chain with countable state space $S = \mathbb{N}$ and transition probability matrix $P = (P_{ij})$. Suppose (V_i) is a right eigenvector for P with eigenvalue λ i.e. for all $i \in \mathbb{N}$,

$$\sum_j P_{ij} V_j = \lambda V_i$$

such that $\mathbb{E}[|V_{X_n}|] < \infty$ for all n. Show that

$$Y_n = \frac{V_{X_n}}{\lambda^n}$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(6) Let Z_n be the population for the n-th generation of the branching process for which each node numbered $i = 1, 2, \ldots Z_n$ independently branches into $X_i^{(n)}$ nodes at the following generation, with mean $\mathbb{E}[X_i^{(n)}] = m > 1$ and variance $\sigma^2 = \mathrm{Var}[X_i^{(n)}]$. Let $Z_0 = 1$.

(a) Compute $\mathbb{E}[Z_n]$. (Hint: Use a conditional expectation to relate $\mathbb{E}[Z_n]$ and $\mathbb{E}[Z_{n+1}]$.

(b) Verify that $\mathbb{E}[Z_{n+1}^2] = m^2 \mathbb{E}[Z_n^2] + \sigma^2 \mathbb{E}[Z_n]$.