Stochastic Processes - Spring 2008

Practice Problems for Final Exam
Bernhard Bodmann, PGH 636
Duration: 150 minutes

First Name: ____________________

Last Name: ____________________

Show all work. No points will be given for numerical answers without working being shown.

(1) Consider the (continuous-time) Poisson process $\{N_t\}_{t\geq 0}$, which has independent increments on disjoint intervals, with distribution given by

$$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}$$

for all $t \geq s \geq 0, k \in \{0, 1, 2, \dots\}$ and a fixed parameter $\lambda > 0$. Show that the process $M_t = N_t - \lambda t$ is a martingale.
Solution Since by the triangle inequality $\mathbb{E}[|M_t|] \leq \mathbb{E}[|N_t|] + \lambda t < \infty$ and increments are independent, we only need to check that the expected value is constant:

$$E[M_t] = \mathbb{E}[N_t] - \lambda t = 0\,.$$

The last equality follows from the definition of the Poisson distribution and from $\mathbb{E}[N_t] = \mathbb{E}[N_t - N_0] = \lambda t$.

(2) Suppose $\{B_t\}_{t\geq 0}$ is a standard Brownian motion starting at $B_0 = 0$. Let $\widetilde{B}_t = tB_1 - B_t$.

(a) Compute $\mathbb{E}[\widetilde{B}_s B_t]$. Give a reason why B_1 is independent of $\sigma(\{\widetilde{B}_s : 0 \leq s \leq 1\})$.

Solution. The new process is a linear combination of Gaussian processes, thus again Gaussian. By definition $\mathbb{E}[\widetilde{B}_t] = \mathbb{E}[tB_1 - B_t] = 0$.

Computing the covariance gives

$$\mathbb{E}[\widetilde{B}_s B_1] = \mathbb{E}[(sB_1 - B_s)B_1] = s\mathbb{E}[B_1 B_1] - \mathbb{E}[B_s B_1] = s - s = 0.$$

Thus, $\widetilde{B}_s$ and B_1 are uncorrelated, and since they are jointly Gaussian, independent. A similar argument works when $\widetilde{B}_s$ is replaced by a vector $\{\widetilde{B}_{s_j}\}_{j=1}^n$ with $0 \leq s_1 < s_2 < \cdots < s_n < 1$.

(b) Compare for fixed $s \in [0,1]$ the distributions of $\widetilde{B}_s$ and $\widetilde{B}_{1-s}$.

Solution. Again, since both $\widetilde{B}_s$ and $\widetilde{B}_{1-s}$ are Gaussian, we only need to compare expectation values and their variances. We obtain $\mathbb{E}[\widetilde{B}_s] = \mathbb{E}[\widetilde{B}_{1-s}] = 0$ and

$$\mathbb{E}[\widetilde{B}_s^2] = s^2\mathbb{E}[B_1^2] - 2s\mathbb{E}[B_1 B_s] + \mathbb{E}[B_s^2] = s - s^2\,.$$

But we have $g(s) = s - s^2 = g(1-s)$, so the variance is the same for $\widetilde{B}_{1-s}$.

(3) Let W_t be a Brownian motion with drift parameter μ, that is $W_t = B_t + \mu t$.

(a) Show that for any real $\lambda > 0$

$$V_t = e^{\lambda W_t - (\lambda\mu + \frac{\lambda^2}{2})t}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(\{W_s : 0 \leq s \leq t\})$.
Solution This is the same as showing $e^{\lambda B_t - \lambda^2 t/2}$ is a martingale. We check the martingale properties directly. Integrability is straightforward by the Gaussian decay of the density of B_t. Consider the canonical filtration $\mathcal{F}_s$.

We have

$$\begin{aligned}\mathbb{E}[e^{\lambda B_t - \lambda^2 t/2} | \mathcal{F}_s] &= \mathbb{E}[e^{\lambda B_s + \lambda(B_t - B_s) - \lambda^2(t-s)/2 + \lambda^2 s/2} | \mathcal{F}_s] \\ &= e^{\lambda B_s - \lambda^2 s/2} \mathbb{E}[e^{\lambda(B_t - B_s) - \lambda^2(t-s)/2}]\end{aligned}$$

In the last step we have extracted the $\mathcal{F}_s$-measurable factor and then used that the increment $B_t - B_s$ is independent of $\mathcal{F}_s$. Now we see

$$\begin{aligned}\mathbb{E}[e^{\lambda(B_t - B_s) - \lambda^2(t-s)/2}] &= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{\infty} e^{\lambda x - \lambda^2 u/2} e^{-x^2/2u}\, dx \\ &= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{\infty} e^{-(x/\sqrt{u} - \lambda\sqrt{u})^2}\, dx = 1\,.\end{aligned}$$

(b) Taking $\lambda = -2\mu$ in (a) you may conclude that $V_t = e^{-2\mu W_t}$ is a martingale. By using a stopping time argument or otherwise show that the probability that the Brownian motion with drift μ reaches $b > 0$ before $a < 0$ is

$$\frac{1 - e^{-2\mu a}}{e^{-2\mu b} - e^{-2\mu a}}\,.$$

Solution We assume that the conditions of Doob's optional stopping theorem hold, so

$$\mathbb{E}[e^{-2\mu W_T}] = \mathbb{E}[e^0] = 1$$

but, calling P_b the desired probability, this expected value is

$$\mathbb{E}[e^{-2\mu W_T}] = P_b e^{-2\mu} + P_a e^{-2\mu a}\,.$$

Now using $P_a = 1 - P_b$ and solving for P_b gives the desired result.

(4) (a) Let B_t denote standard Brownian motion and $\mathcal{F}_t$ the σ-algebra generated by the random variables $\{B_s\}$, $0 \leq s \leq t$.

Let $Y_t = \max 0 \leq s \leq t B_s$. Use the reflection principle to show that for all $t \geq 0$, the distribution of Y_t is identical to that of $|B_t|$.

Solution The reflection principle shows

$$\mathbb{P}(\max_{0\leq s\leq t} B_s \geq \alpha) = 2\mathbb{P}(B_t \geq \alpha)$$

but by the symmetry of Brownian motion this is $2\mathbb{P}(B_t \geq \alpha) = \mathbb{P}(|B_t| \geq \alpha)$.

(b) Let for fixed $t \geq 0$, the family of random variables $\{S_h\}_{0<h<1}$ be

$$S_h = \frac{B_{t+h} - B_t}{h} \, .$$

Show this family has diverging norm in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$\sup_{0<s<1} \mathbb{E}[S_h^2] = \infty \, .$$

Solution The squared norm of S_h is by definition

$$\mathbb{E}[(\frac{B_{t+h} - B_t}{h})^2] = \frac{1}{h^2}\mathbb{E}[(B_{t+h} - B_t)^2] = \frac{1}{h}$$

where we have used that the variance of the increments of BM is equal to the time difference. This last quantity is clearly unbounded.

(5) Suppose that $\{X_n\}$ is a Markov chain with countable state space $S = \mathbb{N}$ and transition probability matrix $P = (P_{ij})$. Suppose (V_i) is a right eigenvector for P with eigenvalue λ i.e. for all $i \in \mathbb{N}$,

$$\sum_j P_{ij} V_j = \lambda V_i$$

such that $\mathbb{E}[|V_{X_n}|] < \infty$ for all n. Show that

$$Y_n = \frac{V_{X_n}}{\lambda^n}$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$
Solution.

To test the martingale property, we have to verify that given $X_n = k$, then the expected value of $V_{X_{n+1}}/\lambda^{n+1}$ is V_k/λ^n.

We know the distribution of X_{n+1} is given by

$$\mathbb{P}(X_{n+1} = i | X_n = k) = P_{ki}\,,$$

so the expected value is by the eigenvalue equation

$$\sum_i P_{ki} V_i / \lambda^{n+1} = V_k / \lambda^n\,.$$

(6) Let Z_n be the population for the n-th generation of the branching process for which each node numbered $i = 1, 2, \ldots Z_n$ independently branches into $X_i^{(n)}$ nodes at the following generation, with mean $\mathbb{E}[X_i^{(n)}] = m > 1$ and variance $\sigma^2 = \mathrm{Var}[X_i^{(n)}]$. Let $Z_0 = 1$.

(a) Compute $\mathbb{E}[Z_n]$. (Hint: Use a conditional expectation to relate $\mathbb{E}[Z_n]$ and $\mathbb{E}[Z_{n+1}]$.
Solution.

Using conditioning on Z_n and $Z_{n+1} = X_1^{(n)} + \ldots X_{Z_n}^{(n)}$, we have

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1} | Z_n]] = m\mathbb{E}[Z_n]\,.$$

Therefore, $\mathbb{E}[Z_n] = m^n$.

(b) Verify that $\mathbb{E}[Z_{n+1}^2] = m^2\mathbb{E}[Z_n^2] + \sigma^2\mathbb{E}[Z_n]$.

Solution.

From the recursion relation for the generating function, $f_n(s) = \mathbb{E}[s^{Z_n}]$, which is $f_{n+1}(s) = f_n(f(s))$, we obtain by differentiating twice

$$\frac{d^2}{ds^2} f_{n+1}(s) = f_n''(f(s))(f'(s))^2 + f_n'(f(s))f''(s)$$

now setting $s = 1$ gives

$$\begin{aligned}\mathbb{E}[Z_{n+1}^2 - Z_{n+1}] &= (\mathbb{E}[Z_n^2] - \mathbb{E}[Z_n])(\mathbb{E}[X])^2 + \mathbb{E}[Z_n]\mathbb{E}[X^2 - X] \\ &= m^2(\mathbb{E}[Z_n^2] - \mathbb{E}[Z_n]) + \mathbb{E}[Z_n](m^2 + \sigma^2 - m) \\ &= m^2\mathbb{E}[Z_n]^2 + (\sigma^2 - m)\mathbb{E}[Z_n]\,.\end{aligned}$$

Now adding the expected value $\mathbb{E}[Z_{n+1}] = m\mathbb{E}[Z_n]$ to both sides gives the desired identity.