Stochastic Processes - Spring 2008

Practice Midterm Exam

First Name: ________________

Last Name: ________________

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Duration: 90 minutes

Show all work. No points will be given for numerical answers without work being shown.

(1) [15 pts] Suppose $X(t)$ is a homogeneous birth-death process with birth-death parameters, $\{\lambda_i, \mu_i\}$, $(i \geq 0)$.

(a) Using standard notation $P_{i,j}(t) = P(X(t) = j|X(0) = i)$, $P_n(t) = P(X(t) = n|X(0) = 0)$ and assuming the standard modeling assumption,

$$P_{i,i} = 1 - (\lambda_i + \mu_i)h + o(h)$$
$$P_{i,i+1} = \lambda_i h + o(h)$$
$$P_{i,i-1} = \mu_i h + o(h)$$

derive differential equations for $P_n(t)$. **Briefly describe the steps of your derivation. You do not need to solve the differential equations**

(b) Let

$$P(t) = (P_{i,j}(t))$$

and define the infinitesimal matrix A by

$$A = \lim_{h \to O^+} \frac{P(h) - I}{h}$$

where I is the identity matrix $I = (\delta_{ij})_{i,j \geq 0}$. Derive a formula for the entries of A in terms of the birth-death parameters.

(2) [20 pts] An airlines reservation system has two computers. An operating computer fails after a period of time which is exponentially distributed with parameter $\mu = 3$. There is one repair facility and repair times are exponentially distributed with parameter $\lambda = 1$. Let $X(t)$ be a random variable giving the number of computers operating at time t, so $X(t)$ has state space $\{0, 1, 2\}$.

(a) Find the infinitesimal matrix A associated to this continuous time Markov process.

(b) Let $P_i(t)$, $0 \leq i \leq 2$, denote the probability that the process is in state i given that $P_2(0) = 1$. Describe how to calculate $P(t) = (P_0(t), P_1(t), P_2(t))$. (You do not need to explicitly determine these probabilitiles).

(c) Find a stationary distribution. Is it unique?

(3) [15 pts] Consider a Yule process $\{X_t\}$ with birth rate λ and $X(0) = 1$ i.e. a pure birth process with $\lambda_n = n\lambda$ and $X(0) = 1$.

(a) Find a system of ordinary differential equations for the probabilities $P_n(t) = \mathbb{P}(X_t = n)$, and appropriate initial conditions.

(b) Derive a differential equation for the expected value $E_t = \mathbb{E}[X_t]$, and solve it to obtain E_t.