(1) $P_n(t) = \mathbb{P}(X_t = n \mid X_0 = 0)$

a) We have for $t, h \geq 0$

$$\begin{aligned} P_n(t+h) &= P_n(t)\, P_{n,n}(h) \\ &\quad + P_{n+1}(t)\, P_{n+1,n}(h) \\ &\quad + P_{n-1}(t)\, P_{n-1,n}(h) + o(h) \end{aligned}$$

where $o(h)$ comes from two or more jumps.

Now, using rates,

$$\begin{aligned} P_n'(t) &= -P_n(t)\,(\lambda_n + \mu_n) \\ &\quad + P_{n+1}(t)\, \mu_{n+1} \\ &\quad + P_{n-1}(t)\, \lambda_{n-1}. \end{aligned}$$

b) $$A = \frac{d}{dh} P(h) \Big|_{h=0}$$

$$A_{ij} = \begin{cases} -(\lambda_i + \mu_i), & i = j \\ \lambda_i, & j = i+1 \\ \mu_i, & j = i-1 \\ 0, & \text{else}, \end{cases}$$

so

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots & \\ & & \cdots & & & \end{pmatrix}.$$

(2) X_t : # comp. operating, $S \in \{0,1,2\}$

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "3"] \arrow[r, bend left, "1"] & 2 \arrow[l, bend left, "6"] \end{tikzcd}$$

birth rate $\lambda = 1$ from 1 repair facility

death rate $\mu_2 = 2\mu_1$, $\mu_1 = \mu$, because "deaths" occur independently

a) $$A = \begin{pmatrix} -1 & 1 & 0 \\ 3 & -4 & 1 \\ 0 & 6 & -6 \end{pmatrix}$$

b) Compute $P(t) = P(0)\, e^{tA}$

c) Stationary dist$^{\underline{n}}$ $A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -4 & 1 \\ 1 & 6 & -6 \end{pmatrix}$,

solve

$$\pi A_1 = (1 \; 0 \; 0)$$

$$\Rightarrow \quad \pi_2 - 6\pi_3 = 0 \quad \Rightarrow \quad \pi_2 = 6\pi_3$$

$$\pi_1 - 4\pi_2 + 6\pi_3 = \pi_1 - 3\pi_2 = 0$$

$$\Rightarrow \quad \pi_1 = 3\pi_2 = 18\pi_3$$

Now use $\pi_1 + \pi_2 + \pi_3 = 18\pi_3 + 6\pi_3 + \pi_3 = 1$

$$\rightarrow \quad \pi_1 = \frac{18}{25}$$

$$\pi_2 = \frac{6}{25}$$

$$\pi_3 = \frac{1}{25}.$$

Unique solution!

(3) a) Initial cond. $P_1(0) = 1$, all other $P_n(0) = 0$.

$$P_n'(t) = -P_n(t)(n\lambda) + P_{n-1}(t)(n-1)\lambda$$

b) $$\mathbb{E}[X_t] = \sum_{n=1}^{\infty} n P_n(t)$$

$$\frac{d}{dt}\mathbb{E}[X_t] = \lambda \sum_{n=1}^{\infty} n\left(-n\cancel{\lambda} P_n(t) + (n-1)P_{n-1}(t)\right)$$

$$= \lambda\Big(- P_1(t) \ast$$

$$- 2^2 P_2(t) + 2 \cdot P_1(t)$$

$$- 3^2 P_3(t) + 3 \cdot 2\, P_2(t) - \ldots \Big)$$

$$= \lambda \left(P_1(t) + 2P_2(t) + 3P_3(t) + \ldots \right)$$

$$= \lambda\, \mathbb{E}[X_t]$$

Solve ODE

$$\Rightarrow \quad \mathbb{E}[X_t] = C e^{\lambda t}, \qquad \mathbb{E}[X_0] = 1$$

$$\overset{\text{init.}}{\Longrightarrow} \quad \mathbb{E}[X_t] = e^{\lambda t}.$$