Stochastic Processes - Spring 2008

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Exercise Sheet 1, due Thursday, February 28, 2008

Do Exercises 1-3 individually. To solve Exercise 4, work in teams of two students.

(1) Let $\{X_t\}$ be a stochastic process giving the size of a population at time t, assume $X_0 = 1$. Suppose that the individuals of the population do not interact and have independently have an individual birth rate λ and an individual death rate μ. By this we mean as usual that each individual gives birth with probability

$$\lambda h + o(h)$$

in a time interval of length h and dies with probability

$$\mu h + o(h)$$

in a time interval of length h. Suppose also that there is a carrying capacity of K individuals for the environment that the population is in so that the population cannot exceed K individuals i.e. that $\lambda_K = 0$. Since $X = 0$ is the only absorbing state and the associated Markov chain is irreducible, the population will eventually die out. Calculate the expected time to extinction. (Problem 19, Page 156, Karlin and Taylor, First Course in Stochastic Processes.)

(2) A system is composed of N identical components each independently operating a random length of time (exponentially distributed with parameter $\tau > 0$) until failure. When a component fails it undergoes repair. The repair time is also random (exponentially distributed with parameter $\rho > 0$). The system is said to be in state n at time t, $X_t = n$, if there are exactly n components undergoing repair at time t. Model this system as a birth-death process i.e. determine the infinitesimal birth-death parameters. (Problem 15, Page 161, Karlin and Taylor, First Course in Stochastic Processes.)

(3) The surface of a bacterium has several sites at which a foreign molecule may become attached if it has an acceptable composition. Consider a par-

ticular site and suppose that molecules arrive at the site according to a Poisson process with parameter $\mu > 0$. Of the molecules that arrive a fraction $0 < \beta < 1$ are acceptable. Unacceptable molecules stay at the site for a length of time exponentially distributed with parameter $\tau > 0$. While at the site they prevent further attachments there. An acceptable molecule remains forever at the site and prevents any further attachments. By setting up the problem as a three-state continuous time Markov chain, find the probability that the site in question does not have an acceptable molecule attached by time t. (Problem 23, Page 163, Karlin and Taylor, First Course in Stochastic Processes. You do not need to express the answer in the form they give, but simplify as much as possible and of course show all work).

(4) An animal population has two species, foxes and rabbits. We denote the number of foxes at time t as F_t and the number of rabbits as R_t. Each birth or death in either species is considered an "occurrence". Assume that in a time interval of length h, more than one occurrence has a probability $o(h)$. Let the rates for the transitions for a length of time h be specified (up to $o(h)$) by constants $\alpha, \beta, \gamma, \delta > 0$:

1. $(r, f) \to (r + 1, f)$ with probability $\alpha rh + o(h)$,
2. $(r, f) \to (r - 1, f)$ with probability $\beta rfh + o(h)$,
3. $(r, f) \to (r, f + 1)$ with probability $\gamma rfh + o(h)$,
4. $(r, f) \to (r, f - 1)$ with probability $\delta fh + o(h)$.

Derive the distribution for waiting times and the conditional probabilities for transitions given that an occurrence takes place at time t. Take note of the special case $r = 0$ or $f = 0$. Use these results to simulate the population dynamics as discussed in class for the choice $\alpha = 1$, $\beta = 0.1$, $\delta = 0.5$, $\gamma = 0.02$, assuming $R_0 = 60$ and $F_0 = 8$. Plot a realization of R_t and F_t as function of t, and plot the path (R_t, F_t) in a two-dimensional coordinate system. Repeat the simulation and comment on the observed properties of the realizations in both plots. Plot the probabilities (density plot) of being in states (r, f) at three different times after starting at $R_0 = 60$ and $F_0 = 8$. Attach a printout of your code in Matlab or in any other standard programming language.