Stochastic Processes - Spring 2008

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Exercise Sheet 1: Sketch of solutions

(1) This is a birth-death process with birth parameter $\lambda_n = n\lambda$ for $n \in \{0, \ldots, K-1\}$ ($\lambda_n = 0$ otherwise) and death parameters $\mu_n = n\mu$. As derived in the lectures, the expected time to absorption in state 0, given the process starts in state 1 is $\sum_{i=1}^{\infty} \rho_i$, where

$$\rho_i = (\prod_{j=1}^{i-1} \lambda_j)/(\prod_{j=1}^{i} \mu_j)\,.$$

Substituting $\lambda_i = i\lambda$ and $\mu_i = i\mu$ we find $\rho_i = \frac{1}{i\lambda}(\frac{\lambda}{\mu})^i$ if $i \leq K$ and $\rho_i = 0$ otherwise. Hence, the expected time to absorption W_1 is

$$W_1 = \frac{1}{\lambda}\sum_{i=1}^{K} \frac{1}{i}(\frac{\lambda}{\mu})^i$$

Realizing that $\sum_{i=0}^{K-1} x^i = \frac{1-x^K}{1-x}$ for $x \neq 1$ and that term-by-term integration gives $\sum_{i=0}^{K-1} \frac{1}{i+1} x^{i+1} = \int_0^x \frac{1-u^K}{1-u} du$, we have the alternative integral expression

$$W_1 = \frac{1}{\lambda}\int_0^{\lambda/\mu} \frac{1-u^K}{1-u} du\,.$$

(2) The components operate independently. If they break, the repair times are also independent. Failure times for operating are exponentially distributed with parameter τ. Repair times are also exponentially distributed with parameter ρ.

Consider the stochastic process $\{X_t\}$ where X_t counts the number of components undergoing repair.

Assume n machines are undergoing repair at the beginning of a time interval $[0, h]$. The probability P of at least one failure occuring in this interval is

$$P = 1 - \mathbb{P}(\text{no failure}) = 1 - \prod_{j=1}^{N-n} \mathbb{P}(j\text{-th unit does not fail}) = 1 - e^{-(N-n)\tau h}$$

Thus, the probability P' of one more failed machine remaining at the end of the interval differs by $o(h)$, which accounts for two or more events occuring, such as two failures and one completed repair, etc. Now taking the derivative at $h = 0$, we get the rate $\lambda_n = (N-n)\tau$.

Similarly, the "death" rate, i.e. of ending up with one machine less undergoing repair, is $\mu_n = n\rho$.

Consequently, the process can be regarded as a birth-death process with parameters $\lambda_n = (N-n)\lambda$ and $\mu_n = n\mu$.

(3) Denote the vacant state by 0, the unacceptable molecule attached by state 1, the acceptable (absorbing) state by 2.

The generator of the stochastic process is

$$A = \begin{pmatrix} -\mu & \mu(1-\beta) & \mu\beta \\ \tau & -\tau & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

In order to compute $P = e^{tA}$, we diagonalize A. A first (left) eigenvector of A is $v_1 = (0,0,1)$ with corresponding eigenvalue $\lambda_1 = 0$, which is the stationary distribution in the absorbing state. The other two eigenvectors are

$$v_2 = \Big(-\frac{\mu+\tau+\Delta}{2\beta\mu}, \frac{\mu(1-2\beta)+\tau+\Delta}{2\beta\mu}, 1\Big)$$

with eigenvalue $\lambda_2 = -\frac{1}{2}(\mu+\tau+\Delta)$ and

$$v_3 = \Big(-\frac{\mu+\tau-\Delta}{2\beta\mu}, \frac{\mu(1-2\beta)+\tau-\Delta}{2\beta\mu}, 1\Big)$$

with eigenvalue $\lambda_3 = -\frac{1}{2}(\mu+\tau-\Delta)$, where $\Delta = \sqrt{(\mu+\tau)^2 - 4\beta\mu\tau}$.

Thus, if we start with an initial distribution

$$\pi = c_1 v_1 + c_2 v_2 + c_3 v_3 ,$$

defined with appropriate coefficients $c_1, c_2, c_3 \in \mathbb{R}$, then after time t, the probability of NOT being in state 2 is

$$\mathbb{P}(\text{not absorbed at time } t) = 1 - (\pi e^{tA})_3 = 1 - c_1 - c_2 e^{\lambda_2 t} - c_3 e^{\lambda_3 t} .$$

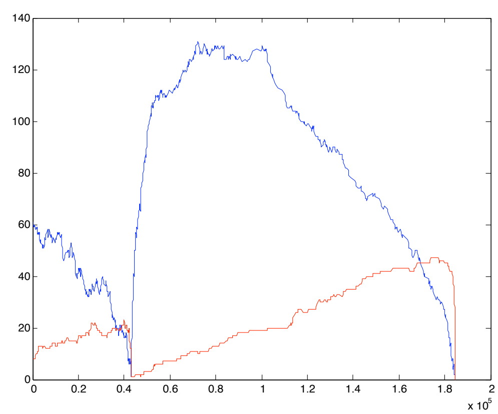
Assuming, as the book does, that we start in state 0, then $c_1 = 1$, $c_2 = \frac{\mu(1-2\beta)+\tau-\Delta}{2\Delta}$ and $c_3 = -\frac{\mu(1-2\beta)+\tau+\Delta}{2\Delta}$.

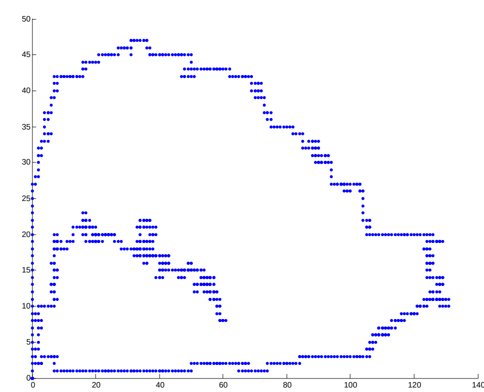
(4) To generate exponentially distributed waiting times with parameter τ, use a r.v. U with uniform distribution in $(0, 1)$ and transform it by $T = -\frac{1}{\tau}\ln(U)$.

These waiting times depend on the state! If there are r rabbits and f foxes, then $\tau = \alpha r + \gamma r f + \beta r f + \delta f$.

At the jump points, we have transition probabilities for the embedded Markov chain. Let $\mu = \alpha r + \beta r f + \gamma r f + \delta f$, then $(r, f) \to (r, f+1)$ with rate $\gamma r f/\mu$, $(r, f) \to (r, f-1)$ with rate $\delta f/\mu$, $(r, f) \to (r+1, f)$ with rate $\alpha r/\mu$, and $(r, f) \to (r-1, f)$ with rate $\alpha r/\mu$,

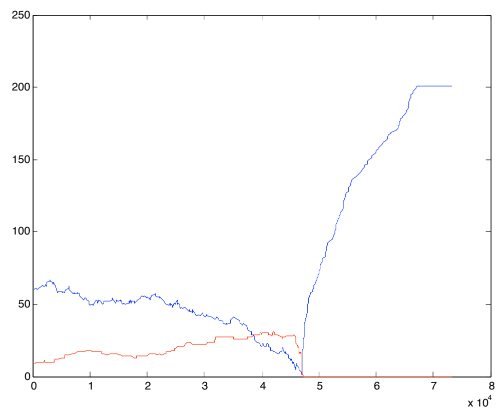
An example plot is given below. (Rabbits: blue, foxes: red)

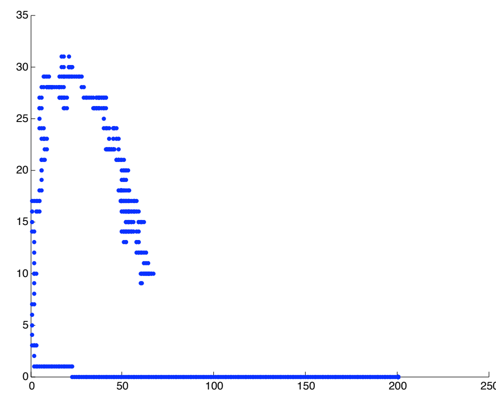




Care must be taken to avoid "rabbit explosions" and also the case $r = 0$ or $f = 0$ needs to be handled separately.

Sometimes the foxes die out and the rabbits are left to multiply. The number of rabbits has been capped at 200.





A nice movie displaying the evolution of the probability distribution in the rabbit-fox state space is due to the group of Sheena Branton, David Johnson, and Robert Rosenbaum, accessible at the website http://www.math.uh.edu/~bgb/Courses/Math6397/rabbitnfox.avi