# Stochastic Processes - Spring 2008

**Bernhard Bodmann, PGH 636**
**Exercise Sheet 2, due Tuesday, April 22, 2008**

**Do all Exercises individually.**

(1) Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p = q$. Let $a, b \in \mathbb{N}$. Define $S_n = X_1 + \ldots + X_n$ and $S_0 = 0$, and let

$$T(\omega) = \inf\{n : S_n(\omega) = -a \text{ or } b\} .$$

Use a stopping time argument to compute the expected value $\mathbb{E}[T]$.

(2) Let $\{X_1, X_2, \ldots\}$ be a sequence of i.i.d. real-valued random variables having finite expectation, and $N$ be a stopping time for the discrete filtration generated by $\{X_n\}_{n=1}^{\infty}$ such that $\mathbb{E}[N] < \infty$.
(a) Why is $\mathbb{E}[X_n 1_{n \le N}] = \mathbb{E}[X_n]\mathbb{P}[\{\omega : N(\omega) \ge n\}]$ for each $n \in \mathbb{N}$?
(b) Using the previous step, prove

$$\mathbb{E}[\sum_{n=1}^{N} X_n] = \mathbb{E}[N]\mathbb{E}[X_1] .$$

(3) Suppose that two candidates run for election. Candidate A obtains $a$ votes and Candidate B obtains $b < a$ votes. Suppose $n = a + b$ is the total number of votes cast. Let $S_k$ be the number of votes by which Candidate A is leading after $k$ votes are counted ($S_k$ can be positive or negative) so that $S_n = a - b$. For $0 \le k \le n - 1$ define

$$X_k = \frac{S_{n-k}}{n-k}$$

(a) Show that $X_0, X_1, \ldots, X_{n-1}$ forms a martingale.
(b) Let

$$T = \min\{k : X_k = 0\}$$

if such a $k$ exists and $T = n - 1$ otherwise. Show that $T$ is a stopping time.

(c) Show that the probability that Candidate $A$ leads throughout the count is $\frac{a-b}{a+b}$.

(4) Let $T$ be the first time a standard Brownian motion crosses the line $l(t) = \alpha + \beta t$, $(\alpha > 0, \beta > 0)$. Determine the characteristic (moment generating) function of $T$ and hence find the expected value of $T$. Hint: Use a martingale method.

(5) Give an expression for the probability that standard Brownian motion, starting at $x = 2$ at time $t = 0$ i.e. $B_0 = 2$, satisfies $B_t < -1$ for some $0 \leq t \leq 3$. Evaluate this expression numerically.

(6) Suppose that $\{B_t\}$ and $\{W_t\}$ are independent standard Brownian motions (starting at zero), and let $\rho \in [0, 1]$ be a constant. Is the process $X_t = \rho B_t + \sqrt{1 - \rho^2} W_t$ a standard Brownian motion? Justify your answer!