Homework 4. Due Wednesday, March 05, 2008

Exercise 4.1. Using identities $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ as well as the chain rule, show that

$$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta,$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin\theta, \quad \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta.$$

Exercise 4.2. Let a and b be some positive numbers such that $b > a > 0$. Sketch the following sets.

- $\Omega_1 = \{(r,\theta) : 0 \le r < a, \quad -\pi \le \theta < \pi\}$
- $\Omega_2 = \{(r,\theta) : a < r < \infty, \quad -\pi \le \theta < \pi\}$
- $\Omega_3 = \{(r,\theta) : a < r < b, \quad -\pi \le \theta < \pi\}$
- $\Omega_4 = \{(r,\theta) : 0 \le r < a, \quad -\pi/4 \le \theta < \pi/4\}$
- $\Omega_5 = \{(r,\theta) : a < r < \infty, \quad -\pi \le \theta < \pi/2\}$
- $\Omega_6 = \{(r,\theta) : a < r < b, \quad \pi/2 \le \theta < 3\pi/2\}$

Remark. In the definition of the polar coordinates we usually use the interval from $-\pi$ to π. Depending on a problem, it is also correct to use any interval $[\alpha, \alpha + 2\pi)$ of length 2π. In particular, in the definition of Ω_6, it is reasonable to use an interval which is a part of $[0; 2\pi)$.

Exercise 4.3. Let n be a positive integer. We define the following functions:

$$\phi_1(r) \equiv 1, \quad \phi_2(r) = \ln(r), \quad \phi_3(r) = r^n, \quad \phi_4(r) = r^{-n}.$$

Determine which function from the list is "physically reasonable" inside Ω_1 (from Exercise 4.2). Explain why. Also explain why the other functions are not "physically reasonable". Repeat this exercise for domains Ω_2 and Ω_3.

Exercise 4.4. Solve Laplace equation outside the quarter-circle of radius one ($r > 1$, $0 < \theta < \pi/2$) subject to the insulated BC on the straight parts of the boundary and the condition $u = f(\theta)$ on the arc of the quarter-circle.

In order to get full credit for this problem, you have to

- Sketch domain Ω, indicate parts of its boundary and the BC on each part. Explain each BC physically (in the context of the heat flow);
- Make complete formulation of the problem. Namely: a) write down the PDE and the range of change of variables where this equation holds; b) on each part of the boundary, write down the BC and the range of change of variables involved; c) write down one additional boundary condition which makes the solution physically reasonable.
- Apply the method of separation of variables and find implied ODEs and BC;
- Analyze three cases ($\lambda < 0$, $\lambda = 0$, and $\lambda > 0$) and find the product form solutions;
- Complete the solution of the problem.

Exercise 4.5. Let Ω be a circular annulus $(1 < r < 2, -\pi \leq \theta < \pi)$ and u be the solution of the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad \Omega, \\ u(1,\theta) &= 1, \quad u(2,\theta) = 2, \quad -\pi \leq \theta < \pi. \end{aligned}$$

a) Sketch Ω. Indicate the BC on the picture;

b) Without solving the above boundary value problem, find the minimum and the maximum values of the the function u; Explain why you got this answer;

c) Without solving the above boundary value problem, explain why the solution function u is circularly symmetric, i.e. it depends on r and does NOT depend on θ;

d) Using answer in part **c)**, solve the boundary value problem. Find the minimum and the maximum values of u and compare them with the values you got in part **b)**.