Homework 10. Due Tuesday, April 24

Exercise 1.

a) Give the definition of the isomorphism between two simple undirected graphs. Tell several words about graph invariants.

b) Extend the definition in **a)** to the case of general undirected graphs, i.e. consider the case when loops and multiple edges are allowed.

c) Define the isomorphism of two general directed graphs.

Solution: **b)**. The bijective function $f : V_1 \to V_2$ has to satisfy the following conditions: (i) There are m loops from point $a \in V_1$ to itself iff there are m loops from point $f(a) \in V_2$ to itself; (ii) There are m different edges between points a and b in V_1 iff there are m different edges between points $f(a)$ and $f(b)$ in V_2.

Exercise 2. See Exercises 36, 42, and 44 from Section 9.3.

Solution:. In problem 36 the graphs are not isomorphic: check the number of vertexes of degree two. In problem 42 the graphs are not isomorphic: each graph has exactly two vertexes of degree four. On the left, they are adjacent while on the right they are not.

Exercise 3. See Exercises 62 and 64 from Section 9.3.

Solution:. In Exercise 62 the graphs are not isomorphic: the vertex u_1 has out-degree two, and the vertex v_3 has out-degree two. Since they are the only vertexes with these properties on the respective graphs, then the isomorphism f, if such exists, has to satisfy $f(u_1) = v_3$. In the similar way, $f(u_2) = v_4$ because these vertexes have in-degree two. We see from the graphs that there is an edge (v_3, v_4) but there is no edge (u_1, u_2).

Exercise 4. Define the concepts of path and circuit for undirected graphs. Which path/circuit is called to be a simple one? Solve Exercise 2 from Section 9.4.

Solution: **a)**. Yes, it is a path. No, this path is not simple, the point b is visited twice. The initial point of the path is not the same as the terminal point, therefore, this path is not a circuit. The length is equal to the number of edges in the path: answer is four; **b)**. This is a circuit. It is not simple: points a and d are visited twice. **c)**. This is not a path, there is no edge (d, b) on the graph. **d)**. This is not a path, there is no edge (b, d) on the graph.

Exercise 5. How can you use the connectivity concept to show whether two graphs are isomorphic or not? Solve Exercises 18 and 20 from Section 9.4.

Solution: Exercise 18. The graphs are not isomorphic. Both graphs have exactly four vertexes of degree three, namely, the vertexes u_1, u_2, u_5, and u_6 on the left graph and v_1, v_3,v_5, and v_7 on the right graph. On the left graph, we can form a circuit u_1, u_2, u_6, u_5, u_1 while on the right graph there are no similar circuits.

Exercise 6. Which undirected graphs are called connected? Define the concept of graph components for undirected graphs. Solve Exercises 3, 4, and 5 from Section 9.4. For each graph in these exercises, find its connected components.

Solution. The graph in Exercise 3 consists of three connected components, in Exercise 4 the graph is connected, and in Exercise 5 the graph consists of two connected components.

Exercise 7.

a) Give the definitions of the connectivity of directed graph.

b) Present an example of a directed graph which is weakly connected but is not strongly connected.

c) Solve Exercise 12, Section 9.4. Also, for each graph, determine its strongly connected components.

Solution of Exercise 12. **a)**. The graph is weakly connected but not strongly connected. It consists of six strong components. **b)**. The circuit a, b, c, d, e, f, a

shows that the graph is strongly connected. **c)**. The underlying undirected graph is not connected, therefore, the graph is not even weakly connected. There is five strongly connected components. Four components consist of a set of one vertex and no edges each. Namely, Let us define $G_1 = (V_1, E_1)$ with $V_1 = \{a\}$ and $E_1 = \emptyset$. This component corresponds to the vertex a. In the similar way, we define the graphs G_2, G_3, and G_4 which correspond to the vertexes g, c, and d, respectively. The fifth strong component is $G_5 = (V_5, E_5)$ with $V_5 = \{b, e, f\}$ and $E_5 = \{(b, f), (e, b), (f, e)\}$.

Exercise 8. What is the difference between a path and a circuit in an undirected graph?

a) Find the powers A^r for the adjacency matrix for the graph K_4 for the cases $r = 1$, $r = 2$, $r = 3$, and $r = 4$;

b) Find the number of circuits of length r in the complete graph K_4 for the cases $r = 1$, $r = 2$, $r = 3$, and $r = 4$;

c) Find the total number of paths of length r in K_4.

a). The adjacency matrix for the complete graph K_4 is a four-by-four matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The powers of this matrix are

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix},$$

and

$$A^4 = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{pmatrix}.$$

b). According to the definition, the circuit is a path with the same initial and terminal vertexes. Therefore, the number of circuits starting from point v and ending at the same point is equal to the diagonal entry, corresponding to the vertex v, of the respective matrix A^r. Therefore, in order to find the number of all circuits of length r, we have to sum up the diagonal entries of A^r (this number is called the trace of A^r). Namely, for $r = 1$, we have no circuits, for $r = 2$, there are 12 circuits, for $r = 3$, there are 24 circuits, and, finally, for $r = 4$, there are 84 circuits.

c). in order to find all paths of length r, we simply have to count the sum of all entries of the matrix A^r.

Exercise 9. Find the city of Königsberg on the world map. What is its modern name? Read wikipedia (www.wikipedia.org) about Leonhard Euler. What is the name of the river in the famous problem "The Seven Bridges of Königsberg"?