Discrete Mathematics. EXAM 1

READ.ME

The test is divided into three major sections: propositional and predicate calculus; proofs; sets and functions. There are many problems but you don't have to solve them all. Most probably you cannot do it simply due time restrictions. The maximum credit for the test you can get is 100 points. The assigned sum of all points for all problems is much higher. So, do not be in a hurry, and, in order to save time during the test, spend five minutes now and read the rules. First, take a quick look to all problems and indicate which of them you can solve fast (or, mark the problems which are too hard to solve fast). Second, keep in mind that the number of credit points you can earn from each particular section does not exceed 40. If, for instance, you get 45, the five extra points will be cut off. Finally, even if you get 40 credit points from each section, you final credit won't exceed 100. Also, there is a couple of tricky questions when you have to do something what is impossible to be done. Just recall a proper definition or a property, write down that you cannot do it and indicate the reason why. Good luck.

1 Propositions and predicates

Problem 1.1

a) Give the definition of a proposition. (**2 pt**)

b) Give an example of a declarative sentence which is not a proposition. Explain. (**2 pt**)

c) Give an example of a proposition which is not a declarative sentence. Explain.(**2 pt**)

d) Give an example of a proposition and determine its truth value. Explain.(**2 pt**)

Problem 1.2 Let p and q be the propositions "I am a criminal" and "I rob banks".

a) Express in simple English the proposition "if p then q" in several different ways. (**3 pt** per variant, **15 pt max**)

b) Express in simple English the proposition which is a contrapositive to "if p then q" (**3 pt**)

Problem 1.3 Let p, q, r, and s be four propositions.

a) Using only the conjunction and negation operators, find a compound proposition which is true iff $p = T$, $q = F$, $r = F$, and $s = T$. (**5 pt**)

b) Using only the negation, conjunction and disjunction operators, find a compound proposition which is true iff exactly three of the propositions p, q, r, and s are true. (**8 pt**)

Problem 1.4 Give an example of a binary predicate $P(x, y)$ when the expression $\forall x \exists y P(x, y)$ is not the same as the expression $\exists y \forall x P(x, y)$. (**2 pt**)

Problem 1.5 Let $F(x, y)$ means "person x can fool person y". Use quantifiers to express the following statements.

a) Peter can fool nobody. (**2 pt**)

b) Everybody can fool himself/herself. (**2 pt**)

c) There is someone who can fool at least two people. (**3 pt**)

d) Everybody can be fooled by exactly two people. (**5 pt**)

Problem 1.6 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\exists x \exists y \forall z P(x, y, z)$;(**2 pt**)

b) $\exists x \exists y P(x, y) \land \forall x \forall y Q(x, y)$. (**4 pt**)

c) $\forall \epsilon > 0 \exists \delta > 0 \big(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon\big)$. (**6 pt**)

2 Proofs

Problem 2.1

a) Give the definition of a valid argument.(**2 pt**)

b) What is a difference between a conjecture and a theorem? Give an example of a conjecture which is a theorem, and an example of a conjecture which is not a theorem. Explain. (**6 pt**)

c) What are the common sources of mistakes in proofs? (**2pt**)

Problem 2.2. Identify an error (errors) in this argument which supposedly shows that if $\forall x \big(P(x) \vee Q(x)\big)$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true. (**10 pt**)

(1)	$\forall x \big(P(x) \vee Q(x)\big)$	Premise
(2)	$P(c) \vee Q(c)$	Universal instantiation from (1)
(3)	$P(c)$	Simplification from (2)
(4)	$\forall x P(x)$	Generalization from (3)
(5)	$Q(c)$	Simplification from (2)
(6)	$\forall x Q(x)$	Generalization from (5)
(7)	$\forall x P(x) \vee \forall x Q(x)$	Conjunction from (4) and (6).

Problem 2.3 Prove that the product of a positive rational number and a positive irrational number is a positive irrational number. What kind of a proof did you use?(**10 pt**)

Problem 2.4 Prove by cases that if x and y are real numbers, then $x + y = \min(x, y) + \max(x, y)$. (**10 pt**)

3 Set theory and functions

Problem 3.1

a) What is a power set for a given set A? Find the power set for set $A = \{0, \sqrt{121}, 11\}$. (**6 pt**)

b) How can you describe the concepts of a universal and the empty sets? (**4 pt**)

c) What a set builder notation? Show the relation between predicates and sets. Which predicate correspond to the empty set and to the universal set? (**6 pt**)

d) Prove one of the de Morgan's laws for sets? (**6 pt**)

Problem 3.2 Let sets A and B consist of at least two elements each. Let f denote some set of assignments of elements in B to some elements in A. Provide simple examples of A, B, and f such that:

a) f is not a function from A to B; (**3 pt**)

b) f is a surjection but not an injection;(**3 pt**)

c) f is an injection but not a surjection; (**3 pt**)

d) f is bijective and f has an inverse; (**3 pt**)

e) f is bijective and f does not have an inverse. (**3 pt**)

Give a brief explanation why it is so in each case.

Problem 3.3 Let A and B be two countable infinite sets, i.e. there exist bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Give an explicit formula of a function which establishes a bijection between A and B, if such exists. Also, give the explicit formula for its inverse. Does it mean that $|A| = |B|$? Explain. (**10 pt**)