Homework 7. Due Wednesday, April 16, 2008

Exercise 7.1.

- Solve Problem **4.4.7** from the book.
- Show that u is a periodic function in time, and find its period, i.e. find a (minimal) number $\tau > 0$ such that $u(x, t + \tau) = u(x, t)$ for all $x \in [0;\ L]$ and all $t > 0$.

Solution. Since $g(x) \equiv 0$, the series form of the solution is

$$u(x,t) \ = \ \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi}{L} n x\right) \cdot \cos\left(\frac{\pi}{L} c t\right)$$

where

$$A_n \ = \ \frac{2}{L} \cdot \int_0^L f(x) \cdot \sin\left(\frac{\pi}{L} n x\right) \text{dx}, \quad n = 1, 2, \dots \ .$$

Using the product of sines formula, we present $u(x,t)$ as the sum of two series

$$u(x,t) \ = \ \frac{1}{2} \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{L} \cdot n \cdot (x - ct)\right) \ + \ \frac{1}{2} \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{L} \cdot n \cdot (x + ct)\right)$$

The sum of the first series is equal to $\frac{1}{2} F(x - ct)$ and the sum of the second series is equal to $\frac{1}{2} F(x + ct)$.

The function F is periodic with period $2L$, i.e. $F(s - 2L) = F(s) = F(s + 2L)$ for all s. Therefore, $u(x,t)$ is periodic in time with period $\frac{2L}{c}$.

Exercise 7.2. The fundamental frequency of a uniform string is given by the formula

$$\omega_1 = \frac{\pi}{L} \cdot \sqrt{\frac{T_0}{\rho_0}}.$$

Q1: What is the meaning of the parameters L, T_0, and ρ_0?

Q2: A guitar player has to tune the strings before the performance. Which parameters stay the same and which parameters change during this "tune-up"?

Q3: A right-handed guitar player makes his performance. He uses both hands to play guitar. What does he do with his right hand? What does he do with his left hand? Which parameter (the only one) changes due to his left-hand actions (consider only one string)? How does the fundamental frequency change due to these actions?

A1: L is the length of string, T_0 is the magnitude of the tensile force, and ρ_0 is the mass density function.

A2: The length of the string remains the same. The mass density function and the tension change.

A3: The right hand causes vibrations of the string. The left hand changes the effective length of the string. Tension and mass density functions remain the same. Since the length decreases from L to some smaller value, the pitch becomes higher.

Exercise 7.3. Solve the problem **4.4.1**, parts **a)** and **b)**.

Solution to part **a)**. The natural frequencies are

$$\omega_n = \frac{\pi}{L} \cdot c \cdot n = \frac{\pi}{L} \cdot \sqrt{\frac{T_0}{\rho_0}} \cdot n \,.$$

Solution to part **b)**. First, we formulate the problem. The formulation is as follows:

$$\begin{array}{ll} \text{(PDE)} & \dfrac{\partial^2 u}{\partial t^2} = c^2 \cdot \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < H, \quad t > 0, \\ \text{(BC)} & u(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(H,t) = 0, \quad t > 0, \\ \text{(IC)} & \text{some initial conditions for } 0 < x < L \end{array}$$

We apply the separation of variables by looking for the solutions of (PDE) satisfying (BC) in the product form and get the following product form solutions (I skip the details)

$$u_n(x,t) = \sqrt{A_n^2 + B_n^2} \cdot \sin(\omega_n t + \theta_n) \cdot \sin\left(\frac{\pi}{H} \cdot (n - \frac{1}{2}) \cdot x\right),$$

with natural frequencies given by

$$\omega_n = \frac{\pi}{H} \cdot \sqrt{\frac{T_0}{\rho_0}} \cdot (n - \frac{1}{2}), \quad n = 1, 2, 3, \dots \,.$$

Exercise 7.4. Let $L = \pi$. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda \cdot \phi(x) = 0, \quad 0 < x < \pi,$$
$$\frac{d\phi}{dx}(0) = 0, \quad \phi(\pi) = 0 \, .$$

Q1. Show that this is a Sturm-Liouville problem. Namely, indicate what are $p = p(x)$ and $\sigma = \sigma(x)$.

Q2. Write down all eigenpairs (λ_n, ϕ_n). *Hint.* You have two choices. Choice 1: derive them by considering three cases: $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. Choice 2: this work has been already done in Exercise **1.6**(Homework 1, Exercise 6). Simply refer to this exercise and write down the formulas.

Q3. Show that the function $\phi_n(x)$ has exactly $n - 1$ zeroes on interval $(0;\ \pi)$. In order to confirm your result, sketch the graphs of the first four eigenfunctions.

Q4. On the same picture, sketch the graphs of the eigenfunctions ϕ_5 and ϕ_6. Mark the zeroes of ϕ_5 as crosses, and the zeroes of ϕ_6 as circles. What can you say about the location of these zeroes with respect to each other?

Q5. Consider the eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \lambda \cdot \sigma(x) \cdot \phi(x) = 0, \quad 0 < x < \pi,$$
$$\frac{d\phi}{dx}(0) = 0, \quad \phi(\pi) = 0 \, .$$

Here, p and σ are arbitrary positive functions. The BC are the same as in the questions **Q1-Q4**. Use your imagination and sketch the first four eigenfunctions of this problem (on the same picture).

A1. $p(x) \equiv 1$ and $\sigma(x) \equiv 1$. Of course, these functions are positive on $0 < x < \pi$.

A2. $\lambda_n = (n - 1/2)^2$, $\phi_n(x) = \cos((n - 1/2)x)$, $n = 1, 2, \ \ldots$.

A3. Let us denote by x_i^n the i-th lowest zero of $\phi_n(x)$. Then we get

$$x_i^n \ = \ \frac{i}{n - 1/2} \cdot \pi, \quad i = 1, 2, \ldots, n-1.$$

Thus, we have exactly $n - 1$ zeroes.

A4. There are 5-1=4 crosses and 6-1 = 5 circles. If we go from the left to the right, they alternate: circle, cross, circle, cross, circle, cross, circle, cross, circle, cross, circle.

A5. The picture has to "resemble" the picture obtained for the case with constant coefficients. The zeroes may be not equally spaced, the amplitudes may be different but the number of zeroes stay the same. For eigenfunctions, corresponding to consecutive eigenvalues, the zeroes have to alternate, etc.