Homework 8. Due Wednesday, April 23, 2008

Exercise 8.1. Consider the following regular Sturm-Liouville problem:

$$\frac{d^2\phi}{dx^2} + \lambda \cdot \phi(x) = 0, \quad 0 < x < 1,$$
$$\frac{d\phi}{dx}(0) = \phi(0), \quad \frac{d\phi}{dx}(1) = \phi(1) .$$

- Find L, $p(x)$, and $\sigma(x)$;
- Show that (λ, ϕ) with $\lambda = -1$ and $\phi(x) = e^x$ is an eigenpair;
- Explain why the negative eigenvalue became possible;
- In the context of the heat flow, explain the meaning of the BC at $x = 0$. Is it physically reasonable? Explain;
- In the context of the heat flow, explain the meaning of the BC at $x = 1$. Is it physically reasonable? Explain.

Solution.

- $L = 1$, $p(x) \equiv 1 > 0$, and $\sigma(x) \equiv 1 > 0$.
- If we plug $\phi(x) = e^x$ and $\lambda = -1$ into the ODE and BC, we get the identities. Therefore, this is an eigenpair.
- Negative sign is possible because at least one of the boundary conditions is not physical.
- If we consider this problem in the context of the heat flow, the Robin boundary conditions originate from the Newton's cooling conditions. In this particular example, at the left endpoint heat flows from hot to cold, which is OK. Meanwhile, at the right endpoint, heat flows from cold to hot, which is, of course, has no physical sense.

Exercise 8.2. Let $L > 0$, and $p = p(x)$ and $\sigma = \sigma(x)$ be positive functions on $0 < x < L$.

a) Write down the ODE which is a part of a regular SL problem;

b) Write down the formula of the $RQ[v]$ for any smooth function v, $v \not\equiv 0$;

c) Write down the boundary conditions for ND problem (Neumann on the left, Dirichlet on the right);

d) Modify the definition of $RQ[v]$ in part **b)** by taking into account the ND boundary conditions;

e) Using Rayleigh quotient, prove that in the case of ND problem we get $\lambda > 0$ for all eigenpairs (λ, ϕ);

f) Repeat the steps **c)-e)** for RN (Robin at the left, Neumann at the right) boundary conditions. You may assume that $h_{\text{left}} = 2008$.

Solutions. **a)** The ODE is

$$\frac{d}{dx}\left(p(x) \cdot \frac{d\phi}{dx}\right) \; + \; \lambda \cdot \sigma \cdot \phi(x), \quad 0 < x < L.$$

b). If $v \not\equiv 0$ then

$$RQ[v] \; = \; \frac{\displaystyle\int_0^L p(x) \left(\frac{d\phi}{dx}\right)^2 \text{dx} \; - \; p(L)\frac{d\phi}{dx}(L)\phi(L) \; + \; p(0)\frac{d\phi}{dx}(0)\phi(0)}{\displaystyle\int_0^L \sigma(x)\phi^2(x)\,\text{dx}}$$

c). The boundary conditions are

$$p(0) \cdot \frac{d\phi}{dx}(0) \; = \; 0 \quad \text{and} \quad \phi(L) \; = \; 0 \; .$$

d). If $v \not\equiv 0$ satisfies the ND boundary conditions, then, in $RQ[v]$, the term $p(0)\frac{d\phi}{dx}(0)\phi(0)$ vanishes due to the Neumann BC, and the term $p(L)\frac{d\phi}{dx}(L)\phi(L)$ vanishes due to the Dirichlet BC. Thus, $RQ[v]$ simplifies to

$$RQ[v] \; = \; \frac{\displaystyle\int_0^L p(x) \left(\frac{d\phi}{dx}\right)^2 \mathrm{dx}}{\displaystyle\int_0^L \sigma(x)\phi^2(x)\,\mathrm{dx}}$$

where the expression on the top is nonnegative and the expression on the bottom is positive.

e). We recall that

$$\lambda \; = \; RQ[\phi].$$

Therefore, $\lambda \geq 0$. If we assume that $\lambda = 0$, then, immidiately, we get the top integral vanishes, i.e.

$$\int_0^L p(x) \left(\frac{d\phi}{dx}\right)^2 \mathrm{dx} \; = \; 0 \; .$$

Since p is strictly positive, it is possible if and only if $\frac{d\phi}{dx} \equiv 0$, or, if and only if ϕ is identically equal to constant. Since $\phi(L) = 0$ due to the Dirichlet BC, we get that the value of this constant is equal to zero. Thus, $\phi \equiv 0$ which contradicts to the fact that ϕ is an eigenfunction. So, we came to contradiction. Therefore, the assumption $\lambda = 0$ is wrong. Thus, $\lambda > 0$

Exercise 8.3. Let $L = \pi$. Consider the vibrations of a perfectly elastic perfectly flexible string with nonuniform mass density $\rho_0 = \rho_0(x)$ and constant magnitude of the tensile force T_0. Suppose that the left end at $x = 0$ is fixed and the right end at $x = \pi$ obeys the elastic boundary condition with the spring constant k. Suppose that the string is initially at rest with a known initial position $f(x)$.

a) Carefully formulate the problem, i.e. write down the PDE, the BC at both ends, and both initial conditions (express the term "initially at rest" mathematically);

b) Apply the first part of the first step of the method of separation of variables. Show details: what form of the solution is sought, how you introduce the separation constant, etc. Write down the t-equation. Write down the ODE and the BC for x-variable. Show that you have the regular SL problem. Namely, write down the formula for $p(x)$ and $\sigma(x)$ and show that they are positive.

c) Write down the Rayleigh quotient (take into account the BC). Show that the minimal eigenvalue λ_1 is positive.

d) Assume that the appropriate eigenvalues and corresponding eigenfunctions are known. Complete the second part of the first step of the method of separation of variables by solving the t-equation and finding product form solutions.

e) Formulate the second step of the method of separation of variables. What weight function are orthogonal to the eigenfunctions? Write down the orthogonality relations and show how to use them in order to find the unknown coefficients.

f) Let ω_n, $n = 1, 2, 3, \ldots$, be natural frequencies of the vibration. Give the formula for ω_n in terms of λ_n. Can you claim that the ratio ω_n/ω_1 is an integer for a general nonuniform string? *Hint.* If there are difficulties to answer this part, please re-read the Section 4.4 (page 145) and read the Section 5.7 (pages 196-197) of the book.

Exercise 8.4. Consider the 1D heat equation with steady sources:

$$c(x)\rho(x)\frac{\partial u}{\partial t} \;=\; \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) \;+\; Q(x), \quad 0 < x < L, \quad t > 0.$$

Write down (formulate only, do not solve) the BV problem (i.e. write down the ODE and BC) for the steady state solution $u_E(x)$ in the case of the following (BC):

DD: $u(0,t) = 30$ C, $u(L,t) = 25$ C, $t > 0$;

ND: $-K_0(0)\frac{\partial u}{\partial x}(0,t) \;=\; -5\ \mathrm{J/(s\cdot m^2)}, \qquad u(L,t) = 32$ C;

RN: $-K_0(0)\frac{\partial u}{\partial x}(0,t) \;=\; -3\cdot(u(0,t) - 32\ \mathrm{C}), \quad -K_0(L)\frac{\partial u}{\partial x}(L,t) \;=\; 0\ \mathrm{J/(s\cdot m^2)}$;

NN: $-K_0(0)\frac{\partial u}{\partial x}(0,t) \;=\; -3\ \mathrm{J/(s\cdot m^2)}, \quad -K_0(L)\frac{\partial u}{\partial x}(L,t) \;=\; 5\ \mathrm{J/(s\cdot m^2)}$.

In your own words, give the physical meaning of each of the boundary conditions. In addition, for the Neumann-Neumann case, derive the conditions on $Q(x)$ when the equilibrium exists.

Hint. Employ the energy conservation law.

Solutions. The ODE does not depend on BC. It is written in the form

$$0 \;=\; \frac{d}{dx}\left(K_0(x)\frac{du_E}{dx}\right) \;+\; Q(x), \quad 0 < x < L,$$

which can be rewritten by

$$\frac{d}{dx}\left(K_0(x)\frac{du_E}{dx}\right) \;=\; -\,Q(x), \quad 0 < x < L.$$

<u>Dirichlet-Dirichlet boundary conditions.</u>

$$u_E(0) = 30\text{ C}, \quad u_E(L) = 25\text{ C}\ .$$

It means that the left end of the rod is heated up (or cooled down, if you spend summer in Houston, TX) to the temperature 86 $F = 30\ C$, and the right end is (definitely cooled down) to the temperature 77 $F = 25\ C$.

<u>Neumann-Dirichlet boundary conditions.</u>

$$-K_0(0)\frac{du_E}{dx}(0) \;=\; -5\text{ J/(s}\cdot\text{m}^2), \qquad u_E(L) = 32\text{ C}\ .$$

The negative sign in Neumann conditions says that the heat flows to the left at the left endpoint, i.e. heat leaks out of the rod. Therefore, every second, we lose 5 Joules on each square meter of the left side of the rod. Meanwhile, the right side of the rod is kept at the constant temperature of 32 degrees C.

<u>Robin-Neumann boundary conditions</u>.

$$-K_0(0)\frac{du_E}{dx}(0) \;=\; -3\cdot(u_E(0) - 32\text{ C}), \quad -K_0(L)\frac{du_E}{dx}(L) \;=\; 0\text{ J/(s}\cdot\text{m}^2)\ .$$

The left endpoint of the rod is attached to the medium with temperature 32 C. The Newton's constant $H_{\text{left}} = 3$ shows the rate how this endpoint cools down. At the right end, there is no exchange of energy, i.e. it is insulated.

Neumann-Neumann boundary conditions.

$$-K_0(0)\frac{du_E}{dx}(0) \;=\; -3 \text{ J/(s} \cdot \text{m}^2), \quad -K_0(L)\frac{du_E}{dx}(L) \;=\; 5 \text{ J/(s} \cdot \text{m}^2).$$

The negative sign at the left indicates that the heat flows to the left, so there is a leaking of heat throught the left boundary at the rate 3 Joules per second on each square meter. The positive sign at the right indicates that the heat flows right, so, again, we have leaking, this, time, at the rate 5 Joules per second on each square meter. In total, the loss rate of energy through each square meter of the boundaries is 11 Joules per second. For equilibrium solutions, we must compensate this amount of heat by some sources inside of rod. Therefore, the condition

$$\int_0^L Q(x)\,\mathrm{dx} \;=\; 11 \text{ J/(s} \cdot \text{m}^2)$$

is necessary and sufficient for the existence of the equilibrium solutions.

Exercise 8.5. Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin(2008x), \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = A, \quad u(\pi,t) = B, \quad t > 0,$$
$$u(x,0) = f(x), \quad 0 < x < \pi;$$

Analyze the $\lim u(x,t)$ as $t \to \infty$.

Solution. The steady state solution solves the BV problem

$$\frac{d^2 u_E}{dx^2} = -\sin(2008x), \quad 0 < x < \pi,$$
$$u_E(0) = A, \quad u_E(\pi) = B.$$

Integrating equation once, we get

$$\frac{du_E}{dx} = c_1 + \frac{1}{2008}\cos(2008x) .$$

Integrating this equation second time, we get

$$u_E(x) = c_2 + c_1 x \cdot \frac{1}{2008^2}\sin(2008x) .$$

Plugging the BC at $x = 0$ and at $x = \pi$, we get the system

$$c_2 = A \quad \text{and} \quad c_2 + c_1\pi = B$$

from which we conclude

$$c_1 = \frac{B - A}{\pi} \quad \text{and} \quad c_2 = A .$$

Therefore,

$$u_E(x) = A + \frac{B - A}{\pi} x + \frac{1}{2008^2}\sin(2008x) .$$

Now, we introduce the displacement of $u(x,t)$ from equilibrium $u_E(x)$ by

$$v(x,t) \;=\; u(x,t) \;-\; u_E(x)\ .$$

Then, $v(x,t)$ is the solution of the following homogeneous problem:

$$(PDE)\quad \frac{\partial v}{\partial t} \;=\; \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$

$$(BC)\quad v(0,t) \;=\; 0, \quad v(\pi,t) \;=\; 0, \quad t > 0,$$

$$(IC)\quad v(x,0) \;=\; f(x) - u_E(x) \;\equiv\; g(x), \quad 0 < x < L\ .$$

Since this problem has no source function ($Q \equiv 0$), and the BC are homogeneous, we easily get the solution in the form of an infinite series

$$v(x,t) \;=\; \sum_{n=1}^{\infty} A_n\, \exp(-n^2 t) \cdot \sin(nx)$$

where the coefficients are defined by

$$A_n \;=\; \frac{2}{\pi}\, g(x)\, \sin(nx)\, \mathrm{dx}, \quad n = 1, 2, 3,\ \ldots\ .$$

Finally, we define u by $u(x,t) = v(x,t) + u_E(x)$. As $t \to \infty$, $u(x,t) \to u_E(x)$.

Exercise 8.6. Give an example (as simple as possible) of a reference temperature distribution $r = r(x,t)$ satisfying the following boundary conditions

$$\textbf{DN:} \quad r(0,t) \;=\; A(t), \quad \frac{\partial r}{\partial x}(L,t) \;=\; B(t);$$

$$\textbf{NN:} \quad \frac{\partial r}{\partial x}(0,t) \;=\; A(t); \quad \frac{\partial r}{\partial x}(L,t) \;=\; B(t);$$

For each of the above BC, compute the reference source function

$$Q_r(x,t) = \frac{\partial r}{\partial t} - \frac{\partial^2 r}{\partial x^2} \; .$$

Hint. For the DD case, look for $r(x,t)$ in the form $r(x,t) = \alpha(t) + \beta(t) \cdot x$ where α and β are some unknown functions. For the NN case, $r(x,t)$ can be found in the form $r(x,t) = \alpha(t) + \beta(t) \cdot x + \gamma(t) \cdot x^2$. Of course, other correct answers are welcome.