# Homework 8. Due Wednesday, April 23, 2008

**Exercise 8.1**. Consider the following regular Sturm-Liouville problem:

$$\frac{d^2\phi}{dx^2} + \lambda \cdot \phi(x) = 0, \quad 0 < x < 1,$$
$$\frac{d\phi}{dx}(0) = \phi(0), \quad \frac{d\phi}{dx}(1) = \phi(1)\ .$$

- Find $L$, $p(x)$, and $\sigma(x)$;
- Show that $(\lambda, \phi)$ with $\lambda = -1$ and $\phi(x) = e^x$ is an eigenpair;
- Explain why the negative eigenvalue became possible;
- In the context of the heat flow, explain the meaning of the BC at $x = 0$. Is it physically reasonable? Explain;
- In the context of the heat flow, explain the meaning of the BC at $x = 1$. Is it physically reasonable? Explain.

**Exercise 8.2**. Let $L > 0$, and $p = p(x)$ and $\sigma = \sigma(x)$ be positive functions on $0 < x < L$.

**a)** Write down the ODE which is a part of a regular SL problem;

**b)** Write down the formula of the $RQ[v]$ for any smooth function $v$, $v \not\equiv 0$;

**c)** Write down the boundary conditions for ND problem (Neumann on the left, Dirichlet on the right);

**d)** Modify the definition of $RQ[v]$ in part **b)** by taking into account the ND boundary conditions;

**e)** Using Rayleigh quotient, prove that in the case of ND problem we get $\lambda > 0$ for all eigenpairs $(\lambda, \phi)$;

**f)** Repeat the steps **c)-e)** for RN (Robin at the left, Neumann at the right) boundary conditions. You may assume that $h_{\text{left}} = 2008$.

**Exercise 8.3**. Let $L = \pi$. Consider the vibrations of a perfectly elastic perfectly flexible string with nonuniform mass density $\rho_0 = \rho_0(x)$ and constant magnitude of the tensile force $T_0$. Suppose that the left end at $x = 0$ is fixed and the right end at $x = \pi$ obeys the elastic boundary condition with the spring constant $k$. Suppose that the string is initially at rest with a known initial position $f(x)$.

**a)** Carefully formulate the problem, i.e. write down the PDE, the BC at both ends, and both initial conditions (express the term "initially at rest" mathematically);

**b)** Apply the first part of the first step of the method of separation of variables. Show details: what form of the solution is sought, how you introduce the separation constant, etc. Write down the $t$-equation. Write down the ODE and the BC for $x$-variable. Show that you have the regular SL problem. Namely, write down the formula for $p(x)$ and $\sigma(x)$ and show that they are positive.

**c)** Write down the Rayleigh quotient (take into account the BC). Show that the minimal eigenvalue $\lambda_1$ is positive.

**d)** Assume that the appropriate eigenvalues and corresponding eigenfunctions are known. Complete the second part of the first step of the method of separation of variables by solving the $t$-equation and finding product form solutions.

**e)** Formulate the second step of the method of separation of variables. What weight function are orthogonal to the eigenfunctions? Write down the orthogonality relations and show how to use them in order to find the unknown coefficients.

**f)** Let $\omega_n$, $n = 1, 2, 3, \ldots$, be natural frequencies of the vibration. Give the formula for $\omega_n$ in terms of $\lambda_n$. Can you claim that the ratio $\omega_n/\omega_1$ is an integer for a general nonuniform string? *Hint.* If there are difficulties to answer this part, please re-read the Section 4.4 (page 145) and read the Section 5.7 (pages 196-197) of the book.

**Exercise 8.4**. Consider the 1D heat equation with steady sources:

$$c(x)\rho(x)\frac{\partial u}{\partial t} \;=\; \frac{\partial u}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) \;+\; Q(x), \quad 0 < x < L, \quad t > 0.$$

Write down (formulate only, do not solve) the BV problem (i.e. write down the ODE and BC) for the steady state solution $u_E(x)$ in the case of the following (BC):

**DD:** $u(0,t) = 30$ C, $u(L,t) = 25$ C, $t > 0$;

**ND:** $-K_0(0)\frac{\partial u}{\partial x}(0,t) \;=\; -5$ J/(s $\cdot$ m$^2$), $u(L,t) = 32$ C;

**RN:** $-K_0(0)\frac{\partial u}{\partial x}(0,t) \;=\; -3\cdot(u(0,t) - 32$ C), $-K_0(L)\frac{\partial u}{\partial x}(L,t) \;=\; 0$ J/(s $\cdot$ m$^2$);

**NN:** $-K_0(0)\frac{\partial u}{\partial x}(0,t) \;=\; -3$ J/(s $\cdot$ m$^2$), $-K_0(L)\frac{\partial u}{\partial x}(L,t) \;=\; 5$ J/(s $\cdot$ m$^2$).

In your own words, give the physical meaning of each of the boundary conditions. In addition, for the Neumann-Neumann case, derive the conditions on $Q(x)$ when the equilibrium exists.

*Hint*. Employ the energy conservation law.

**Exercise 8.5**. Solve the 1D heat equation

$$\frac{\partial u}{\partial t} \;=\; \frac{\partial^2 u}{\partial x^2} \;+\; \sin(2008x), \quad 0 < x < \pi, \quad t > 0,$$

$$u(0,t) = A, \quad u(\pi,t) = B, \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 < x < \pi;$$

Analyze the $\lim u(x,t)$ as $t \to \infty$.

**Exercise 8.6**. Give an example (as simple as possible) of a reference temperature distribution $r = r(x,t)$ satisfying the following boundary conditions

$$\textbf{DN:}\quad r(0,t) \;=\; A(t), \quad \frac{\partial r}{\partial x}(L,t) \;=\; B(t);$$

$$\textbf{NN:}\quad \frac{\partial r}{\partial x}(0,t) \;=\; A(t); \quad \frac{\partial r}{\partial x}(L,t) \;=\; B(t);$$

For each of the above BC, compute the reference source function

$$Q_r(x,t) = \frac{\partial r}{\partial t} - \frac{\partial^2 r}{\partial x^2} \ .$$

*Hint.* For the DD case, look for $r(x,t)$ in the form $r(x,t) = \alpha(t) + \beta(t) \cdot x$ where $\alpha$ and $\beta$ are some unknown functions. For the NN case, $r(x,t)$ can be found in the form $r(x,t) = \alpha(t) + \beta(t) \cdot x + \gamma(t) \cdot x^2$. Of course, other correct answers are welcome.