Test 02, Monday, March 31
Before you start to solve the test, read carefully the instructions

- Quickly read the formulation of all problems.
- Today's test has 6 (SIX) problems.
- For each problem, estimate the ratio "number of points"/ "time to solve", define the optimal order in which you feel more comfortable to solve the problems.
- I DO NOT require the problems to be solved in the order Problem 1, Problem 2, Problem 3, etc. Rather, I HIGHLY RECOMMEND you to solve the problems in your personal OPTIMAL ORDER.
- If you have questions about formulations of problems, DO NOT waste your time guessing what I want from you. RAISE your hand and ask.
- If you are stuck solving a particular problem, leave some space in your blue book, and proceed to solve another problem. If the time permits you can always return to that problem later.
- Carefully read each problem and do what I ask you to do. No points for solving wrong formulated problem.
- If you are done before 10:50, do not forget to recheck your solutions.
- If your solution is incomplete you get partial credit.
- GOOD LUCK

Exercise 1. Fourier series graph sketching. [20 pt].

- Let $L = 1$. For the function $f(x) = 1 - x$, sketch the graphs of its Fourier series, Fourier cosine series, and Fourier sine series on a segment $(-3;\ 3)$.
- Write down in your own words the steps which you have to complete in order to sketch the graph of the Fourier sine series.

Exercise 2. Odd and even functions, and Fourier series.[25 pt]

Let $f = f(x)$ be a function defined in an interval symmetric w.r.t. origin.

- When f is called an odd function? What can you say about the graph of an odd function? Sketch a sample graph.
- When f is called an even function? What can you say about the graph of an even function? Sketch a sample graph.
- For the function
$$f(x) \ = \ e^x + x^{2008} - 3363$$
find its odd part $f_O = f_O(x)$ and even part $f_E = f_E(x)$. *Hint.* Find the even and odd parts of the exponential function first.
- For the function $f(x) = e^x + x^{2008} - 3363$ defined in $-\pi < x < \pi$ write down the formulas for its Fourier coefficients. These formulas have to involve only the integration over the one-sided segment $0 < x < \pi$. DO NOT COMPUTE these integrals.

Exercise 3. **Term-by-term differentiation of the Fourier series.** [30 pt]

Let $L = \pi$ and

$$\sinh(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} a_n \, \cos(nx)$$

be the Fourier cosine series of the hyperbolic sine function.

- Sketch the graph of the hyperbolic sine function on interval $(0; \pi)$;
- Sketch the graph of the above series on interval $(-3\pi; 3\pi)$.
- Can you "carelessly" differentiate a Fourier cosine series term by term? A Fourier sine series? If not, what do you have to take into account?
- Using the identity $\dfrac{d^2}{dx^2}\Big[\sinh(x)\Big] \;=\; \sinh(x)$ derive the formulas for $a_0, a_1, a_2, \ldots$, by CORRECTLY differentiating the above series twice.

Supporting formulas

If $f = f(x)$ is a continuous function on $(0;\ L)$ such that both f and f' are piecewise smooth, then, CORRECTLY differentiating the Fourier sine series

$$f(x) \;\sim\; \sum_{n=1}^{\infty} b_n \, \sin\left(\frac{\pi}{L} nx\right)$$

we get the Fourier cosine series

$$f'(x) \;\sim\; A_0 \;+\; \sum_{n=1}^{\infty} A_n \, \cos\left(\frac{\pi}{L} nx\right)$$

with

$$A_0 = \frac{1}{L}\Big[f(L) - f(0)\Big]$$

and

$$A_n \;=\; \frac{2}{L}\Big[(-1)^n f(L) - f(0)\Big] + \frac{\pi}{L} \cdot n \cdot b_n, \quad n = 1, 2, \ldots \;.$$

Exercise 4. Laplace's equation in polar coordinates.[25 pt]

Consider the following boundary value problem in Ω:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \ + \ \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \ &= \ 0, \quad 0 < r < 1, \quad 0 < \theta < \frac{\pi}{3} \\
\frac{\partial u}{\partial \theta}(r,0) \ = \ 0, \quad \frac{\partial u}{\partial \theta}(r,\tfrac{\pi}{3}) \ &= \ 0, \quad 0 < r < 1, \\
u(1,\theta) \ &= \ f(\theta), \quad 0 < \theta < \frac{\pi}{3}
\end{aligned}$$

- Sketch Ω and indicate on the picture the BC on each part of the boundary;
- Complete the formulation of the problem by adding one more BC which makes the solution physically reasonable;
- Derive the ODEs and the corresponding BCs using the method of separation of variables.

Exercise 5. **Qualitative properties of Laplace's equation** [10 pt]

a) What is the direction of the heat flow?

b) Let $u = u(x, y, t)$ be a solution function of a homogeneous 2D heat equation (i.e. there are no sinks/sources). Assume that (x_0, y_0) be an interior point of a region Ω, such that

$$u(x_0, y_0, t_0) \;=\; \max_{(x,y)\in\Omega} u(x, y, t_0) \;\equiv\; M(t_0)\ .$$

What can you say about the value $u(x_0, y_0, t)$ when t is slightly larger than t_0?

c) Using the answer in **b)** explain physically why a non-constant solution of the Laplace equation cannot attain its maximum at an interior point.

Exercise 6. **1D wave equation** [10 pt]

a) Which natural phenomenon is modeled by 1D wave equation?

b) Which force is responsible for the stretching of a string? What can you say about the direction and the magnitude of this force in the case of a perfectly flexible perfectly elastic string?

c) Which physical law is used to derive the 1D wave equation?

d) Write down the 1D wave equation.