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Analysis of a One-Dimensional Free Boundary Flow Problem

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Summary A one-dimensional free surface problem is considered. It consists in Burgers' equation with an additional diffusion term on a moving interval. The well-posedness of the problem is investigated and existence and uniqueness results are obtained locally in time. A semi-discretization in space with a piecewise linear finite element method is considered. *A priori* and *a posteriori* error estimates are given for the semi-discretization in space. A time splitting scheme allows to obtain numerical results in agreement with the theoretical investigations.

Key words Burgers' equation, Free boundary problem, Small time existence, Finite element method, A priori and a posteriori estimates

Mathematics Subject Classification (1991): 65M12, 65M60, 35R35

1 Introduction

Free surfaces often appear in the frame of fluid flow problems, see *e.g.* [15], or fluid-structure interaction problems, see [7]. In this article we investigate a one-dimensional free surface problem which consists in the motion of a fluid of particles on a space interval with one free extremity. This problem is a simplification in a one space-dimension of the Navier-Stokes equations with a free surface encountered in [13].

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Such a model involves the velocity u of the liquid and the position s of the free right end side of the space interval. The velocity is assumed to satisfy the viscous Burgers equation on the free space interval, while the position of the free extremity is determined by an ordinary differential equation depending on u. Similar problems have already been treated from the theoretical point of view in the literature, for instance for an Arbitrary-Lagrangian-Eulerian formulation in [5], for a two-fluids problem in [1] or for a given moving boundary in [2].

The aim of this article is to investigate this one-dimensional free surface problem from both the theoretical and the numerical points of view in order to prove the well-posedness of our problem and to obtain error estimates for the semi-discretization in space. The results presented in this article are extracted from [3], where they are completed by further numerical investigations.

Our model is presented in Sect. 2. First the function s describing the right abscissa of the free interval is assumed to be given and the problem with given boundary is investigated in Sect. 3. The solvability of such a problem is obtained with a Faedo-Galerkin method and a fixed point theorem. The free boundary problem is then investigated in Sect. 4 by using another fixed point theorem. In Sect. 5, a semi-discretization in space is considered by using a finite element method. The solvability of the semi-discretized problem and error estimates are investigated. *A priori* error estimates are obtained by using a variational method, see *e.g.* [8]. *A posteriori* error estimates are also obtained, see [16]. Numerical experiments are then discussed in Sect. 6. A time splitting scheme decouples the advection and diffusion phenomena. The advection part is solved with an Arbitrary Lagrangian-Eulerian procedure and an upwind finite differences scheme, while the diffusion part is solved with an implicit finite element method. Numerical experiments are made and the results are in agreement with the theoretical investigations.

2 Modeling

A one-dimensional free surface problem is presented. It consists in Burgers' equation with an additional diffusion term on a space interval with one free extremity. Let $T > 0$ be the final time of the simulation. Let $s : (0,T) \to \mathbb{R}$ be an unknown positive function defined on the interval $(0,T)$ and let the space-time domain Q_T be defined by:

$$Q_T = \{(y,t) : y \in (0,s(t)), t \in (0,T)\} .$$

Given the real positive numbers s_0, α and ε and the functions $\bar{f} : \mathbb{R} \times (0, T) \to \mathbb{R}$ and $\bar{u}_0 : (0, s_0) \to \mathbb{R}$, our problem is to find $s : (0, T) \to \mathbb{R}$ and $u : Q_T \to \mathbb{R}$ satisfying:

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial y} - \varepsilon \frac{\partial^2 u}{\partial y^2} = \bar{f},\ (y,t) \in Q_T, \tag{1}$$

$$u(y,0) = \bar{u}_0(y),\ \ y \in (0, s_0), \tag{2}$$

$$u(0,t) = 0,\ \ t \in (0,T), \tag{3}$$

$$\frac{\partial u}{\partial y}(s(t),t) = 0,\ \ t \in (0,T), \tag{4}$$

$$\dot{s}(t) = u(s(t),t),\ \ t \in (0,T), \tag{5}$$

$$s(0) = s_0, \tag{6}$$

where $\dot{s}(t) = \frac{d}{dt}s(t)$. Equations (1)-(4) are the Burgers equation with mixed Dirichlet-Neumann boundary conditions. Equations (5)-(6) are the coupling equations, necessary to determine the free surface $s(t)$ and to ensure the well-posedness of the mathematical problem.

The problem (1)-(6) in the non-cylindrical domain Q_T is turned into a problem in a cylindrical domain, namely $U_T = (0,1) \times (0,T)$ by using the change of variables $x = y/s(t)$ or $y = s(t)x$. The functions f, u_0 and $\tilde{u}$ are defined respectively by $f(x,t) = \bar{f}(s(t)x,t)$, $u_0(x) = \bar{u}_0(s_0 x)$ and $\tilde{u}(x,t) = u(s(t)x,t)$, $0 < x < 1$, $0 < t < T$. For the sake of simplicity, the function $\tilde{u}$ is denoted again by u. The problem of interest is then to find $s : (0,T) \to \mathbb{R}$ and $u : U_T \to \mathbb{R}$ satisfying:

$$\frac{\partial u}{\partial t} + \frac{\alpha}{s(t)} u \frac{\partial u}{\partial x} - \frac{\varepsilon}{s(t)^2} \frac{\partial^2 u}{\partial x^2} - \frac{\dot{s}(t)}{s(t)} x \frac{\partial u}{\partial x} = f,\ (x,t) \in U_T, \tag{7}$$

$$u(x,0) = u_0(x),\ \ x \in (0,1), \tag{8}$$

$$u(0,t) = 0,\ \ t \in (0,T), \tag{9}$$

$$\frac{\partial u}{\partial x}(1,t) = 0,\ \ t \in (0,T), \tag{10}$$

$$\dot{s}(t) = u(1,t),\ \ t \in (0,T), \tag{11}$$

$$s(0) = s_0. \tag{12}$$

The first objective is to prove that, for sufficiently regular data, there exists a final time $0 < \hat{T} \leq T$ such that (7)-(12) has a unique solution (u, s) defined for $t \in (0, \hat{T})$. Then *a priori* and *a posteriori* error estimates are obtained for a semi-discretization in space using a finite element method. A time splitting scheme allows finally to obtain numerical results.

3 Given Boundary Problem

In order to investigate the problem (7)-(12), a linear problem is considered first. The nonlinear term appearing in the Burgers equation and the additional equation $\dot{s}(t) = u(1, t)$ will then be taken into account by using successively two fixed point techniques.

Let the space V be defined by

$$V = \left\{ v \in H^1(0,1), v(0) = 0 \right\},$$

equipped with semi-norm $|v|_V := |v|_{H^1(0,1)} = ||v'||_{L^2(0,1)}$ and norm $||v||_V := ||v||_{H^1(0,1)} = \sqrt{||v||^2_{L^2(0,1)} + ||v'||^2_{L^2(0,1)}}$. We can also observe that, since $v(0) = 0$ in V, the norm $||v||_V$ and semi-norm $|v|_V$ are equivalent. The dual space of V is denoted by V'. In the following it is assumed that $f \in L^2(U_T)$, $u_0 \in V$ and $s_0 > 0$ are given.

In the whole article, C and C_i will denote generic constants and $(\cdot, \cdot)$ and $||\cdot||$ will denote respectively the $L^2(0,1)$-scalar product and $L^2(0,1)$-norm.

Let β_1 and β_2 be two positive constants such that $\beta_1 < s_0 < \beta_2$. Let s be a function of $W^{1,\infty}(0,T)$ such that $\beta_1 \leq s(t) \leq \beta_2$ for all $t \in (0,T)$. Let $g \in L^2(U_T)$ and $w_0 \in V$ be two given functions. The following general linear problem is considered first: find $w : U_T \to \mathbb{R}$ satisfying:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\varepsilon}{s(t)^2} \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\dot{s}(t)}{s(t)} x \dfrac{\partial w}{\partial x} = g, \ (x,t) \in U_T, \\ w(0) = w_0, \qquad x \in (0,1), \\ w(0,t) = \dfrac{\partial w}{\partial x}(1,t) = 0, \qquad t \in (0,T). \end{cases} \tag{13}$$

Lemma 1 *Problem (13) has a unique solution w. Furthermore $w \in L^2(0,T,H^2(0,1) \cap V) \cap H^1(0,T,L^2(0,1))$ and there exist two constants D_1 and D_2 independent of g, w_0, s and T, such that:*

$$||w||^2_{L^2(0,T,H^2(0,1)) \cap H^1(0,T,L^2(0,1))} \leq D_1(1 + R^2 T) e^{D_2 R^2 T} \left(||g||^2_{L^2(U_T)} + |w_0|^2_V \right), \tag{14}$$

where $R = \max_{t\in(0,T)} |\dot{s}(t)|$.

Proof In order to obtain *a priori* estimates, let us assume that $w \in L^2(0,T,V) \cap H^1(0,T,V')$ satisfies for all v in V:

$$\int_0^1 \frac{\partial w}{\partial t} v dx + \frac{\varepsilon}{s(t)^2} \int_0^1 \frac{\partial w}{\partial x}\frac{\partial v}{\partial x} dx - \frac{\dot{s}(t)}{s(t)} \int_0^1 x \frac{\partial w}{\partial x} v dx = \int_0^1 g v dx, \quad (15)$$

which is the weak formulation of (13). By taking $v = w$ in (15), we obtain:

$$\frac{1}{2}\frac{d}{dt}||w||^2 + \frac{\varepsilon}{\beta_2^2}\left|\left|\frac{\partial w}{\partial x}\right|\right|^2 \leq ||g||\,||w|| + \frac{R}{\beta_1}\left|\left|\frac{\partial w}{\partial x}\right|\right|\,||w||\,.$$

By using Young and Poincaré inequalities, this relationship leads to:

$$\frac{1}{2}\frac{d}{dt}||w||^2 + \frac{\varepsilon}{2\beta_2^2}\left|\left|\frac{\partial w}{\partial x}\right|\right|^2 \leq \frac{c_p^2\beta_2^2}{\varepsilon}||g||^2 + \frac{R^2\beta_2^2}{\beta_1^2\varepsilon}||w||^2, \quad (16)$$

where c_p is the constant appearing in the Poincaré inequality. Gronwall's lemma permits to conclude first that:

$$||w(t)||^2 \leq e^{CR^2T}\left(C\int_0^T ||g||^2 d\tau + ||w_0||^2\right), \text{ a.e. } t \in (0,T).$$

Then, by integration of (16) on $(0,T)$, we obtain by using the previous relationship:

$$\begin{aligned}\left|\left|\frac{\partial w}{\partial x}\right|\right|^2_{L^2(U_T)} &\leq C\int_0^T ||g||^2 d\tau + ||w_0||^2 + CR^2T\,||w||^2_{L^\infty(0,T,L^2(0,1))} \\ &\leq e^{CR^2T}(1+R^2T)\left(C\int_0^T ||g||^2 d\tau + ||w_0||^2\right).\end{aligned}$$

By taking $v = -\frac{\partial^2 w}{\partial x^2}$ in (15), similar arguments lead to (14).

Now a Faedo-Galerkin method is used (see [12] for instance) to discretize (15) in the space variable. That means we are looking for a Galerkin approximation w_n of w in the finite dimensional space V_n of V given by $V_n = \text{span}\left\{\sin\left(\left(l+\frac{1}{2}\right)\pi x\right) : 1 \leq l \leq n\right\}$. Since (15) is a linear problem, it is easy to see that w_n is a solution of a linear system of ordinary differential equations and hence does exist. So, by using a priori estimates (14), we can prove there exists a subsequence of $(w_n)_{n=1}^{\infty}$ which tends to w, solution of (15), when n tends to infinity. The regularity and uniqueness of the solution are obvious. □

Let $\sigma : (g, w_0) \to w$ be the linear operator which maps the right-hand sides (g, w_0) of (13) into w. Let K be a positive constant such that

$$K > 2\sqrt{D_1}\, |u_0|_V \tag{17}$$

and let $T > 0$ be such that

$$D_1(1 + K^2T)e^{D_2K^2T}\, |u_0|_V^2 \leq K^2/4. \tag{18}$$

The set $\mathcal{S}(T)$ of acceptable boundary functions is defined by:

$$\begin{aligned} \mathcal{S}(T) = \big\{ s \in W^{1,\infty}(0,T)\ :\ & \beta_1 \leq s(t) \leq \beta_2, \quad \forall t \in [0,T], \\ & s(0) = s_0 \quad \text{and} \quad |\dot{s}(t)| \leq K, \quad \text{a.e. } t \in [0,T] \big\}. \end{aligned} \tag{19}$$

Let the function s be fixed in $\mathcal{S}(T)$. Our auxiliary nonlinear problem is then to find $u_s : U_T \to \mathbb{R}$ satisfying

$$\begin{cases} \dfrac{\partial u_s}{\partial t} + \dfrac{\alpha}{s(t)} u_s \dfrac{\partial u_s}{\partial x} - \dfrac{\varepsilon}{s(t)^2}\dfrac{\partial^2 u_s}{\partial x^2} - \dfrac{\dot{s}(t)}{s(t)} x \dfrac{\partial u_s}{\partial x} = f, & (x,t) \in U_T, \\ u_s(x,0) = u_0(x), & x \in (0,1), \\ u_s(0,t) = 0, & t \in (0,T), \\ \dfrac{\partial u_s}{\partial x}(1,t) = 0, & t \in (0,T). \end{cases} \tag{20}$$

The notation u_s shows that the solution of (20) depends on the given function $s \in \mathcal{S}(T)$.

Theorem 1 *There exists a time $0 < \tilde{T} \leq T$ such that, for all $s \in \mathcal{S}(T)$, problem (20) has a unique solution u_s. Furthermore $u_s \in L^2(0, \tilde{T}, H^2(0,1) \cap V) \cap H^1(0, \tilde{T}, L^2(0,1))$ and there exists a constant $\tilde{K} = \tilde{K}(\tilde{T})$, independent of s such that:*

$$||u_s||_{L^2(0,\tilde{T},H^2(0,1)) \cap H^1(0,\tilde{T},L^2(0,1))} \leq 2\tilde{K} \left\{ ||f||_{L^2(U_T)} + |u_0|_V \right\}. \tag{21}$$

Proof Let us observe that the following continuous embedding holds (see [9] for instance):

$$L^2(0,T,H^2(0,1)) \cap H^1(0,T,L^2(0,1)) \hookrightarrow C^0\left(\overline{U_T}\right) \tag{22}$$

and define the functional spaces $W_1(T)$ and $W_2(T)$ by

$$\begin{aligned}
W_1(T) &= \left\{ v \in L^2(0,T,H^2(0,1) \cap V) \cap H^1(0,T,L^2(0,1)) \right\}, \\
W_2(T) &= \left\{ v \in L^2(0,T,V) \cap H^1(0,T,V') \right\}.
\end{aligned}$$

Taking into account (22), we can define the continuous operator $\tilde{\tau}$ by:

$$\begin{aligned}
\tilde{\tau}: \quad & W_1(T) \to W_2(T), \\
& u_s(x,t) \to u_s(x,t)\frac{\partial u_s}{\partial x}(x,t) = \frac{1}{2}\frac{\partial}{\partial x}u_s^2(x,t).
\end{aligned}$$

With this notation, problem (20) is equivalent to

$$\begin{cases}
\dfrac{\partial u_s}{\partial t} - \dfrac{\varepsilon}{s(t)^2}\dfrac{\partial^2 u_s}{\partial x^2} - \dfrac{\dot{s}(t)}{s(t)}x\dfrac{\partial u_s}{\partial x} = g, & (x,t) \in U_T, \\
u_s(x,0) = u_0(x), & x \in (0,1), \\
u_s(0,t) = 0, & t \in (0,T), \\
\dfrac{\partial u_s}{\partial x}(1,t) = 0, & t \in (0,T),
\end{cases} \tag{23}$$

with $g(x,t) := f(x,t) - \dfrac{\alpha}{s(t)}\tilde{\tau}(u_s(x,t))$. It follows that, if $\tau : W_1(T) \to W_1(T)$ is defined by

$$\tau(u) = \sigma\left(f - \frac{\alpha}{s}\tilde{\tau}(u), u_0\right),$$

then the solution of (23) is a fixed point of τ.

Since the embedding $W_2(T) \hookrightarrow L^2(U_T)$ is compact (see [12] for instance) and by using Lemma 1, we can see that the operator τ is continuous and compact. In order to apply the Schauder fixed point theorem, we have to look for *a priori* estimates. Let $\tilde{K}$ be the constant defined by

$$\tilde{K} = \sqrt{D_1(1 + K^2T)e^{D_2K^2T}}, \tag{24}$$

where D_1, D_2 are the constants introduced in (14) and K, T are the constants defined in (17) (18). Let $L > 0$ be the real positive number given by

$$L = 2\tilde{K}\left\{||f||_{L^2(U_T)} + |u_0|_V\right\} \tag{25}$$

and $W(T)$ the set defined by $W(T) = \left\{v \in W_1(T) : ||v||_{W_1(T)} \leq L\right\}$.

Since $s \in \mathcal{S}(T)$ and by using Lemma 1 and (25), we have, if $\tilde{u} = \tau(u)$, with $u \in W(T)$:

$$\begin{aligned} ||\tilde{u}||_{W_1(T)} &\leq \tilde{K} \left\{ \left|\left| f - \frac{\alpha}{s(t)} u \frac{\partial u}{\partial x} \right|\right|_{L^2(U_T)} + |u_0|_V \right\} \\ &\leq \frac{L}{2} + \tilde{K} \frac{\alpha}{\beta_1} \left|\left| u \frac{\partial u}{\partial x} \right|\right|_{L^2(U_T)}. \end{aligned}$$

By using (22), this last term can be bounded by:

$$\begin{aligned} \tilde{K} \frac{\alpha}{\beta_1} ||u||_{L^\infty(U_T)} \left|\left| \frac{\partial u}{\partial x} \right|\right|_{L^2(U_T)} &\leq \tilde{K} \frac{\alpha}{\beta_1} ||u||_{L^\infty(U_T)} ||u||_{L^\infty(0,T,V)} T^{1/2} \\ &\leq \tilde{K} \frac{\alpha}{\beta_1} L^2 T^{1/2}. \qquad (26) \end{aligned}$$

Hence there exists $0 < \tilde{T} \leq T$ such that $\tilde{K} \frac{\alpha}{\beta_1} L^2 \tilde{T}^{1/2} \leq \frac{L}{2}$ and $||\tilde{u}||_{W_1(\tilde{T})} \leq L$. The Schauder fixed point theorem (see for instance [6]) permits to obtain the existence of a fixed point of the restriction of τ to $W(\tilde{T})$ since $\tau(W(\tilde{T})) \subset W(\tilde{T})$.

The uniqueness of the solution is proved with a variational argument. Let u_1 and u_2 be two solutions to (20) in $W(\tilde{T})$ and let the difference $u_1 - u_2$ be denoted by w. The function w satisfies, for all v in V and almost every t in $(0,T)$:

$$\begin{aligned} \int_0^1 \frac{\partial w}{\partial t} v dx - \frac{\dot{s}(t)}{s(t)} \int_0^1 x \frac{\partial w}{\partial x} v dx + \frac{\varepsilon}{s(t)^2} \int_0^1 \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} dx \\ = -\frac{\alpha}{s(t)} \int_0^1 \left(u_1 \frac{\partial u_1}{\partial x} - u_2 \frac{\partial u_2}{\partial x} \right) v dx, \end{aligned}$$

with initial condition $w(0) = 0$. This relationship gives, with $v = w$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} ||w||^2_{L^2(0,1)} + \frac{\varepsilon}{s(t)^2} \left|\left| \frac{\partial w}{\partial x} \right|\right|^2_{L^2(0,1)} &\leq \frac{|\dot{s}(t)|}{s(t)} \int_0^1 x \left| \frac{\partial w}{\partial x} \right| |w| \, dx \\ &+ \frac{\alpha}{s(t)} \int_0^1 \left| u_1 \frac{\partial u_1}{\partial x} - u_2 \frac{\partial u_2}{\partial x} \right| |w| \, dx. \qquad (27) \end{aligned}$$

By using the embedding $H^1(0,1) \hookrightarrow C^0([0,1])$ and Poincaré inequality, it is easy to show that the nonlinear term in (27) satisfies:

$$
\begin{aligned}
&\int_0^1 \left| u_1 \frac{\partial u_1}{\partial x} - u_2 \frac{\partial u_2}{\partial x} \right| |w| \, dx \\
&\leq C_1 \left(\left|\left| \frac{\partial u_1}{\partial x} \right|\right|_{L^2(0,1)} + \|u_2\|_{C^0([0,1])} \right) ||w||_{L^2(0,1)} \left|\left| \frac{\partial w}{\partial x} \right|\right|_{L^2(0,1)} \\
&\leq 2C_1 L \, ||w||_{L^2(0,1)} \left|\left| \frac{\partial w}{\partial x} \right|\right|_{L^2(0,1)},
\end{aligned}
$$

where C_1 is a constant depending only on the domain $(0, 1)$. Finally Young inequalities and the fact that $s \in \mathcal{S}(T)$ are used to show that there exists a constant C_2 such that (27) leads to:

$$
\frac{d}{dt} \|w\|^2_{L^2(0,1)} + \frac{\varepsilon}{2\beta_2^2} \left|\left| \frac{\partial w}{\partial x} \right|\right|^2_{L^2(0,1)} \leq C_2 \, ||w||^2_{L^2(0,1)} .
$$

Gronwall's lemma allows us to conclude that $w = 0$. $\square$

4 Free boundary problem

The problem with a free boundary (7)-(12) is now investigated. The set $\mathcal{S}(T)$ defined by (19) is convex and closed in $W^{1,\infty}(0, T)$. There exists $0 < \tilde{T} \leq T$ such that, for each $s \in \mathcal{S}(T)$, Problem (20) admits a unique solution $u_s \in W_1(\tilde{T})$. For each $s \in \mathcal{S}(T)$, let $\tilde{s} : (0, \tilde{T}) \to \mathbb{R}$ be defined by:

$$
\tilde{s}(t) = s_0 + \int_0^t u_s(1, \tau) d\tau. \tag{28}
$$

Theorem 1 and (22) imply that (28) is well-defined. The following lemma is first proved:

Lemma 2 *There exists a time* $0 < \bar{T} \leq \tilde{T}$ *such that* $\tilde{s} \in \mathcal{S}(\bar{T})$ *when* $s \in \mathcal{S}(T)$.

Proof Clearly $\tilde{s}(0) = s_0$ and since the function $u_s(1, t)$ is continuous in time, we have by using Theorem 1:

$$
\begin{aligned}
|\tilde{s}(t)| &\leq s_0 + \int_0^t |u_s(1, \tau)| \, d\tau \leq s_0 + t \, ||u_s(1)||_{C^0([0,\tilde{T}])} \\
&\leq s_0 + \tilde{T} \, ||u_s||_{W_1(\tilde{T})} \leq s_0 + \tilde{T} 2\tilde{K} \left(||f||_{L^2(U_T)} + |u_0|_V \right),
\end{aligned}
$$

for $0 < t \leq \tilde{T}$. The assumption $s_0 \in (\beta_1, \beta_2)$ implies that $\tilde{s}(t) \in (\beta_1, \beta_2)$ when $t \leq \bar{T}$ is small enough. As $\frac{d}{dt}\tilde{s}(t) = u_s(1,t)$, we obtain by using again Theorem 1:

$$\left|\dot{\tilde{s}}(t)\right| = |u_s(1,t)| \leq ||u_s||_{W_1(\tilde{T})} \leq 2\tilde{K}\left(||f||_{L^2(U_T)} + |u_0|_V\right).$$

It suffices to recall the definition (17) (18) of K and take $\bar{T}$ sufficiently small to prove that $\left|\dot{\tilde{s}}(t)\right| \leq K$ for $0 < t \leq \tilde{T}$ and conclusion follows. □

In the following, the time $\bar{T}$ is denoted again by T and the following application is considered:

$$\begin{aligned} \gamma : \mathcal{S}(T) &\rightarrow \mathcal{S}(T); \\ s \quad &\rightarrow \tilde{s} \quad \text{defined by (28).} \end{aligned}$$

In order to prove the existence and uniqueness of a solution to the problem (7)-(12), we have to show that γ has a unique fixed point. In fact, if u_s is the solution to the problem (20) associated with s such that $\dot{s}(t) = u_s(1,t)$, the couple (u_s, s) is the unique solution to (7)-(12).

Theorem 2 *There exists* $0 < \hat{T} \leq T$ *such that* $\gamma : \mathcal{S}(\hat{T}) \rightarrow \mathcal{S}(\hat{T})$ *possesses a unique fixed point* s.

Proof According to Lemma 2, the range of γ is included in $\mathcal{S}(T)$. Let $s \in \mathcal{S}(T)$ and u_s be the solution to (20). The operator γ can be written:

$$\begin{tikzcd}
\mathcal{S}(T) \arrow[r] & W_1(T) \arrow[r, hook, "c."] & C^0(\overline{U_T}) \arrow[r] & C^0([0,T]) \arrow[r] & \mathcal{S}(T) \\
s(t) \arrow[r] & u_s(x,t) \arrow[r, hook, "c."] & u_s(x,t) \arrow[r] & u_s(1,t) \arrow[r] & \tilde{s}(t),
\end{tikzcd}$$

where $\tilde{s}(t) = s_0 + \int_0^t u_s(1,\tau)d\tau$. In order to prove that γ is continuous and compact, it suffices to show that the application which maps s into u_s is continuous. Consider a sequence $(s_n)_{n=0}^{\infty}$ strongly convergent in $\mathcal{S}(T)$, *i.e.*

$$s_n \rightarrow s \text{ in } \mathcal{S}(T) \text{ strongly, according to } ||\cdot||_{W^{1,\infty}(0,T)}. \tag{29}$$

A function $u_n \in W_1(T)$ is associated with each term s_n of this sequence, defined by the solution to (20) with given boundary s_n. The sequence $(u_n)_{n=0}^{\infty}$ is uniformly bounded in $W_1(T)$ by a constant which does not depend on the sequence $(s_n)_{n=0}^{\infty}$ (see Theorem 1).

The spaces $H^1(0,T,L^2(0,1))$ and $L^2(0,T,H^2(0,1))$ being reflexives and by taking (22) into account, there exist subsequences, also denoted by $(u_n)_{n=0}^{\infty}$, satisfying:

$$(u_n) \rightharpoonup u \text{ in } L^2(0,T,H^2(0,1)) \text{ weakly },\tag{30}$$

$$\left(\frac{\partial u_n}{\partial t}\right) \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(0,T,L^2(0,1)) \text{ weakly and}\tag{31}$$

$$(u_n) \rightarrow u \text{ in } C^0\left(\overline{U_T}\right) \text{ strongly }.\tag{32}$$

Since $u_n(1,\cdot)$ converges strongly to $u(1,\cdot)$ in $C^0([0,T])$, then $\tilde{s}_n$ defined by $\tilde{s}_n(t) = s_0 + \int_0^t u_n(1,\tau)d\tau$ converges strongly to $\tilde{s}(t) = s_0 + \int_0^t u(1,\tau)d\tau$ in $W^{1,\infty}(0,T)$.

In order to show that γ is continuous, we have to prove now that u is the solution to (20) associated to s. Our definition of u_n from s_n implies that (u_n, s_n) satisfies:

$$\int_{U_T} \frac{\partial u_n}{\partial t} v dx dt + \int_{U_T} \frac{\varepsilon}{s_n(t)^2} \frac{\partial u_n}{\partial x} \frac{\partial v}{\partial x} dx dt - \int_{U_T} \frac{\dot{s}_n(t)}{s_n(t)} x \frac{\partial u_n}{\partial x} v dx dt$$
$$+ \int_{U_T} \frac{\alpha}{s_n(t)} u_n \frac{\partial u_n}{\partial x} v dx dt = \int_{U_T} f v dx dt, \qquad \forall v \in L^2(0,T,V).$$

By taking the limit in this last relationship (*i.e.* by using (29)-(32)), the limit function u is the unique solution to (20). It follows that $s \in \mathcal{S}(T) \rightarrow u_s \in W_1(T)$ is continuous and so γ is a continuous compact operator.

The Schauder fixed point theorem together with Lemma 2 allows us to conclude that application γ has one fixed point in $\mathcal{S}(T)$ which is denoted in the following by $s(t)$. For this function s, we have $\dot{s}(t) = u_s(1,t)$, $0 < t < T$.

In order to prove the uniqueness of the solution of (7)-(12), γ is proved to be a contraction from $\mathcal{S}(T)$ into $\mathcal{S}(T)$ for sufficiently small T. The complete proof is given in [3], in which elementary estimates allow us to obtain

$$||\tilde{s}_1 - \tilde{s}_2||_{W^{1,\infty}(0,T)} \leq E(T)\, ||s_1 - s_2||_{W^{1,\infty}(0,T)}$$

and $E(T)$ tends to zero when T tends to zero. □

In order to conclude this paragraph, let us remark that the existence and uniqueness of a solution to (7)-(12) is obtained locally in time for all positive bounded values of the coefficient α contrary to [5] where α was fixed to 3/2.

Finally, the solution (u, s) to the free surface problem (7)-(12) can be transferred on the original domain $Q_{\hat{T}}$ to obtain the existence and uniqueness of the original problem (1)-(6), which is well-posed at least for small times.

5 Semi-Discretization in Space and Error Estimates

In this section, a semi-discretization in space is introduced using a finite element method and the existence and uniqueness of the solution to the semi-discrete problem is proved. In a second step, *a priori* error estimates are obtained, as in [8] for instance. Finally an *a posteriori* error analysis is investigated, as in [16] for instance.

A weak form of the continuous problem (7)-(12) consists in finding (u, s) satisfying:

$$\begin{aligned}
&\int_0^1 \frac{\partial u}{\partial t} v dx + \frac{\alpha}{s(t)} \int_0^1 u \frac{\partial u}{\partial x} v dx - \frac{\dot{s}(t)}{s(t)} \int_0^1 x \frac{\partial u}{\partial x} v dx \qquad (33) \\
&+ \frac{\varepsilon}{s(t)^2} \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = \int_0^1 f v dx, \quad \forall v \in V, \quad \text{a.e. } t \in (0, T), \\
&u(x, 0) = u_0(x), \quad x \in (0, 1), \\
&\dot{s}(t) = u(1, t), \quad \text{a.e. } t \in (0, T) \quad \text{with } s(0) = s_0.
\end{aligned}$$

A piecewise linear finite element semi-discretization in space of (33) is introduced. Let $N \in \mathbb{N}$ be a given integer and let $x_0 = 0 < x_1 < x_2 < \ldots < x_{N+1} = 1$ be the discretization points in the interval $[0, 1]$. In the following, we set $h_i = x_{i+1} - x_i$ and $h = \max_i h_i$. The space of continuous functions on $[0, 1]$ which are vanishing into $x = 0$ and whose restrictions on $[x_i, x_{i+1}]$, $i = 0, \ldots, N$ belong to $\mathbb{P}_1$ is denoted by V_h.

Let $u_{0,h}$ be the interpolant of $u_0 \in V$ in V_h. We have $|u_{0,h}|_V \leq |u_0|_V$, and, if $u_0 \in H^2(0, 1)$, interpolation results (see for instance [4]) give rise to

$$||u_0 - u_{0,h}||_{L^2(0,1)} \leq Ch^2 |u_0|_{H^2(0,1)}, \tag{34}$$

where C is and will be in the following a generic constant independent of h.

Let $s_h \in \mathcal{S}(T)$ be an approximation of $s \in \mathcal{S}(T)$. The semi-discretization in space of (33) consists in finding $u_h : (0, T) \to u_h(t) \in V_h$ and $s_h \in \mathcal{S}(T)$ satisfying, for all v_h in V_h:

$$\int_0^1 \frac{\partial u_h}{\partial t} v_h dx + \frac{\alpha}{s_h(t)} \int_0^1 u_h \frac{\partial u_h}{\partial x} v_h dx - \frac{\dot{s}_h(t)}{s_h(t)} \int_0^1 x \frac{\partial u_h}{\partial x} v_h dx$$

$$+\frac{\varepsilon}{s_h(t)^2} \int_0^1 \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} dx = \int_0^1 f_h v_h dx, \text{ a.e. } t \in (0,T) \tag{35}$$

$$u_h(x,0) = u_{0,h}(x), \quad x \in (0,1),$$

$$\dot{s}_h(t) = u_h(1,t), \quad t \in (0,T), \quad \text{with } s_h(0) = s_0.$$

Let us observe that the semi-discretization (35) of (33) corresponds, by setting $y_j(t) = s_h(t)x_j$ and by changing the variable x into y, to the spatial semi-discretization of the weak formulation (1)-(6) by using a piecewise linear finite element method on the moving grid $y_0(t), y_1(t), \ldots, y_{N+1}(t)$. Later this problem will be discretized in time by using an Arbitrary Lagrangian-Eulerian method, which is well adapted to this framework.

Let us mention that the spatial variable can be changed (as we have done for obtaining (7)-(12) from (1)-(6)) by setting $f_h(x,t) = \bar{f}(s_h(t)x,t)$, where $\bar{f}$ is the right-hand side of (1). With this change of variables, $||f_h||_{L^2(U_T)}$ is bounded by $C_S \left|\left|\bar{f}\right|\right|_{L^2(Q_T)}$ where C_S is a constant independent of s_h and h. From now on, $\bar{f}$ is assumed to belong to $L^2(0,T,H^1(0,\beta_2))$ in order to estimate later the difference between f and f_h.

Theorem 3 *There exists* $0 < \hat{T} \leq T$ *(*$\hat{T}$ *independent of* $h > 0$*) such that (35) has a unique solution* $(u_h, s_h) \in H^1(0,\hat{T},V_h) \times \mathcal{S}(\hat{T})$, $\forall h > 0$. *Furthermore, there exists a constant* C *independent of* h *such that:*

$$||u_h||_{H^1(0,\hat{T},V_h)} \leq C \left\{ \left|\left|\bar{f}\right|\right|_{L^2(Q_{\hat{T}})} + |u_0|_V \right\}. \tag{36}$$

Proof The steps of the proof are the same as those of the proofs of Theorems 1 and 2. The difference consists in getting a solution u_h in $H^1(0,\hat{T},V_h)$ and not in $L^2(0,T,H^2(0,1))$ since piecewise linear finite elements are used. Finally observe that the time interval $(0,\hat{T})$, on which the solution u_h is defined, is independent of h due to the uniform boundedness of s_h in $\mathcal{S}(\hat{T})$. □

A priori error estimates are proved in the next theorem.

Theorem 4 (A priori error estimates) *Assume that* $u_0 \in H^2(0,1)$ *with* $u_0 \in V$ *and* $\bar{f} \in L^2(0,T,H^1(0,\beta_2))$. *There exists* $0 < \tilde{T} \leq T$, *independent of* h, *such that (33), respectively (35), has a unique solution* (u, s), *respectively* (u_h, s_h), $\forall h > 0$, *and the following error estimate holds:*

$$||u - u_h||_{L^\infty(0,\tilde{T},L^2(0,1))\cap L^2(0,\tilde{T},V)} + ||s - s_h||_{H^1(0,\tilde{T})} \leq Ch, \qquad (37)$$

where the constant C *does not depend on* h.

Proof Theorems 2 and 3 ensure the existence of a time $0 < \tilde{T} \leq T$ independent of $h > 0$ such that u and u_h are well-defined on $(0, \tilde{T})$, $\forall h > 0$. In the following, the final time $\tilde{T}$ is denoted again by T and the error is defined by $e = u - u_h$.

By subtracting (35) from (33), we obtain for all $\tilde{v}_h$ in V_h and almost every t in $(0, T)$:

$$\begin{aligned}
&\left(\frac{\partial e}{\partial t}, \tilde{v}_h\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial e}{\partial x}, \frac{\partial \tilde{v}_h}{\partial x}\right) = \frac{\dot{s}(t)}{s(t)}\int_0^1 x\frac{\partial e}{\partial x}\tilde{v}_h dx \qquad (38)\\
&+\left(\frac{\dot{s}(t)}{s(t)} - \frac{\dot{s}_h(t)}{s_h(t)}\right)\int_0^1 x\frac{\partial u_h}{\partial x}\tilde{v}_h dx - \left(\frac{\varepsilon}{s(t)^2} - \frac{\varepsilon}{s_h(t)^2}\right)\int_0^1 \frac{\partial u_h}{\partial x}\frac{\partial \tilde{v}_h}{\partial x}dx\\
&-\alpha\int_0^1\left(\frac{u}{s(t)}\frac{\partial u}{\partial x} - \frac{u_h}{s_h(t)}\frac{\partial u_h}{\partial x}\right)\tilde{v}_h dx + (f - f_h, \tilde{v}_h)\,.
\end{aligned}$$

We immediately see that, for all v_h in V_h:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,||e||^2 + \frac{\varepsilon}{\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \leq \frac{1}{2}\frac{d}{dt}\,||e||^2 + \frac{\varepsilon}{s(t)^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2\\
&= \left(\frac{\partial e}{\partial t}, u - u_h\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial e}{\partial x}, \frac{\partial(u - u_h)}{\partial x}\right)\\
&= \left(\frac{\partial e}{\partial t}, u - v_h\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial e}{\partial x}, \frac{\partial(u - v_h)}{\partial x}\right)\\
&+ \left(\frac{\partial e}{\partial t}, v_h - u_h\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial e}{\partial x}, \frac{\partial(v_h - u_h)}{\partial x}\right). \qquad (39)
\end{aligned}$$

Let us consider $0 < t \leq T$. The last two terms of (39) are transformed by using (38) with $\tilde{v}_h = v_h - u_h$ to obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|e\right|\right|^2+\frac{\varepsilon}{\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \leq \left|\left|\frac{\partial e}{\partial t}\right|\right|\left|\left|u-v_h\right|\right|+\frac{\varepsilon}{\beta_1^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|\left|\left|\frac{\partial (u-v_h)}{\partial x}\right|\right| \\
&+\frac{\dot{s}(t)}{s(t)}\int_0^1 x\frac{\partial e}{\partial x}(v_h-u_h)dx+\left(\frac{\dot{s}(t)}{s(t)}-\frac{\dot{s}_h(t)}{s_h(t)}\right)\int_0^1 x\frac{\partial u_h}{\partial x}(v_h-u_h)dx \\
&-\left(\frac{\varepsilon}{s(t)^2}-\frac{\varepsilon}{s_h(t)^2}\right)\int_0^1 \frac{\partial u_h}{\partial x}\frac{\partial (v_h-u_h)}{\partial x}dx \qquad (40) \\
&-\alpha\int_0^1\left(\frac{u}{s(t)}\frac{\partial u}{\partial x}-\frac{u_h}{s_h(t)}\frac{\partial u_h}{\partial x}\right)(v_h-u_h)dx+(f-f_h,v_h-u_h)\,.
\end{aligned}
$$

The test function $\tilde{v}_h = v_h - u_h$ is decomposed into $v_h - u + u - u_h = (v_h - u) + e$. By using several Cauchy-Schwarz inequalities and (21) (36), there exist some constants $C_1, C_2, \ldots$, independent of h and t such that relationship (40) becomes:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|e\right|\right|^2+\frac{\varepsilon}{\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \leq \quad \left|\left|\frac{\partial e}{\partial t}\right|\right|\left|\left|u-v_h\right|\right|+C_1\left|\left|\frac{\partial e}{\partial x}\right|\right|\left|\left|\frac{\partial (u-v_h)}{\partial x}\right|\right| \\
&+C_2\left|\left|\frac{\partial e}{\partial x}\right|\right|\left|\left|u-v_h\right|\right|+C_3\left|\left|\frac{\partial e}{\partial x}\right|\right|\left|\left|e\right|\right|+C_4\left|\frac{\dot{s}(t)}{s(t)}-\frac{\dot{s}_h(t)}{s_h(t)}\right|\left|\left|u-v_h\right|\right| \\
&+C_5\left|\frac{\dot{s}(t)}{s(t)}-\frac{\dot{s}_h(t)}{s_h(t)}\right|\left|\left|e\right|\right|+C_6\left|s(t)-s_h(t)\right|\left|\left|\frac{\partial (u-v_h)}{\partial x}\right|\right| \qquad (41) \\
&+C_7\left|s(t)-s_h(t)\right|\left|\left|\frac{\partial e}{\partial x}\right|\right|+C_8\left|\left|\frac{u}{s(t)}\frac{\partial u}{\partial x}-\frac{u_h}{s_h(t)}\frac{\partial u_h}{\partial x}\right|\right|\left|\left|u-v_h\right|\right| \\
&+C_9\left|\left|\frac{u}{s(t)}\frac{\partial u}{\partial x}-\frac{u_h}{s_h(t)}\frac{\partial u_h}{\partial x}\right|\right|\left|\left|e\right|\right|+\left|\left|f-f_h\right|\right|\left|\left|u-v_h\right|\right|+\left|\left|f-f_h\right|\right|\left|\left|e\right|\right|\,.
\end{aligned}
$$

The following relations hold, with constants C_i, $i = 1, \ldots, 6$, independent of h and t:

$$
\left|\frac{\dot{s}(t)}{s(t)}-\frac{\dot{s}_h(t)}{s_h(t)}\right| \leq C_1\left|s(t)-s_h(t)\right|+C_2\left|\dot{s}(t)-\dot{s}_h(t)\right|, \qquad (42)
$$

$$
\begin{aligned}
&\left|\left|\frac{u}{s(t)}\frac{\partial u}{\partial x}-\frac{u_h}{s_h(t)}\frac{\partial u_h}{\partial x}\right|\right| \leq C_3\left(\left|\left|u\right|\right|_{L^\infty(0,1)}\left|\left|\frac{\partial e}{\partial x}\right|\right|\right. \\
&\left.+\left|\left|\frac{\partial u_h}{\partial x}\right|\right|\left|\left|e\right|\right|_{L^\infty(0,1)}+\left|s(t)-s_h(t)\right|\left|\left|u_h\frac{\partial u_h}{\partial x}\right|\right|\right)\,.
\end{aligned}
$$

By using Poincaré inequality, (21) and (36), we obtain

$$\begin{aligned}\left|\left|\frac{u}{s(t)}\frac{\partial u}{\partial x}-\frac{u_h}{s_h(t)}\frac{\partial u_h}{\partial x}\right|\right| &\leq C_4\left(\left|\left|\frac{\partial u}{\partial x}\right|\right|+\left|\left|\frac{\partial u_h}{\partial x}\right|\right|\right)\left|\left|\frac{\partial e}{\partial x}\right|\right| \\ +C_5\left|\left|\frac{\partial u_h}{\partial x}\right|\right|^2 |s(t)-s_h(t)| &\leq C_6\left(\left|\left|\frac{\partial e}{\partial x}\right|\right|+|s(t)-s_h(t)|\right).\end{aligned} \tag{43}$$

We consider $v_h = R_h u$ in relation (41), where $R_h u$ is the Lagrange interpolant of u. Interpolation results (see [4, volume 2] for instance) and relations (42)-(43) imply that there exist constants $C_1, \ldots, C_8$, independent of h and t, such that (41) becomes by using Poincaré inequalities:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||e||^2+\frac{\varepsilon}{\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \leq \left|\left|\frac{\partial e}{\partial t}\right|\right| C_1 h^2 ||u||_{H^2(0,1)} \\ &+C_2\left|\left|\frac{\partial e}{\partial x}\right|\right| h\,||u||_{H^2(0,1)}+C_3\left|\left|\frac{\partial e}{\partial x}\right|\right| ||e|| + C_4|\dot{s}-\dot{s}_h| h^2 ||u||_{H^2(0,1)} \\ &+C_5|s-s_h| h\,||u||_{H^2(0,1)}+C_6|\dot{s}-\dot{s}_h|\,||e|| \\ &+C_7|s-s_h|\left|\left|\frac{\partial e}{\partial x}\right|\right| + C_8||f-f_h||\,h^2||u||_{H^2(0,1)}+||f-f_h||\,||e||.\end{aligned} \tag{44}$$

By observing that $\dot{s}(t)-\dot{s}_h(t)=u(1,t)-u_h(1,t)=\int_0^1 \frac{\partial e}{\partial x}(\xi,t)d\xi$, we have:

$$|\dot{s}-\dot{s}_h| \leq \left|\left|\frac{\partial e}{\partial x}\right|\right| \tag{45}$$

and, using a Young inequality:

$$\frac{d}{dt}\left(|s-s_h|^2\right) \leq \left|\left|\frac{\partial e}{\partial x}\right|\right|^2 + |s-s_h|^2. \tag{46}$$

Recall that the functions f, f_h are respectively defined by $f(x,t) = \bar{f}(s(t)x,t)$ and $f_h(x,t) = \bar{f}(s_h(t)x,t)$, $x \in (0,1)$ and that $\bar{f}$ is assumed to be in $L^2(0,T,H^1(0,\beta_2))$. It is easy to notice that

$$f(x,t)-f_h(x,t)=\int_0^1 \frac{\partial \bar{f}}{\partial x}(\tau s(t)x+(1-\tau)s_h(t)x)\cdot x(s(t)-s_h(t))d\tau.$$

Therefore, there exists a constant C such that:

$$||f-f_h||^2 \leq C\left|\bar{f}\right|_{H^1}^2 |s(t)-s_h(t)|^2, \tag{47}$$

where $\left|\bar{f}\right|_{H^1}^2$ belongs to $L^1(0,T)$. By using (45), (47) and Young inequalities, there exist constants C_i, $i=1,2,3$ such that relationship (44) gives:

$$\frac{d}{dt}\left|\left|e\right|\right|^2 + \frac{\varepsilon}{2\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \leq C_1 h^2\left(\left|\left|u\right|\right|_{H^2(0,1)}\left|\left|\frac{\partial e}{\partial t}\right|\right| + \left|\left|u\right|\right|_{H^2(0,1)}^2\right)$$
$$+(C_2 + C\left|\bar{f}\right|_{H^1}^2)\left|s-s_h\right|^2 + C_3\left|\left|e\right|\right|^2. \tag{48}$$

Inserting relationship (46) into (48) implies that:

$$\frac{d}{dt}\left|\left|e\right|\right|^2 + \frac{\varepsilon}{2\beta_2^2}\frac{d}{dt}\left(\left|s-s_h\right|^2\right) \leq C_1 h^2\left(\left|\left|u\right|\right|_{H^2(0,1)}\left|\left|\frac{\partial e}{\partial t}\right|\right| + \left|\left|u\right|\right|_{H^2(0,1)}^2\right)$$
$$+(C_2 + K\left|\bar{f}\right|_{H^1}^2 + \frac{\varepsilon}{2\beta_2^2})\left|s-s_h\right|^2 + C_3\left|\left|e\right|\right|^2. \tag{49}$$

Notice that $\left|\left|u\right|\right|_{H^2(0,1)}\left|\left|\frac{\partial e}{\partial t}\right|\right| \leq \frac{1}{2}\left|\left|u\right|\right|_{H^2(0,1)}^2 + \frac{1}{2}\left|\left|\frac{\partial e}{\partial t}\right|\right|^2 \in L^1(0,T)$ and recall that $u \in L^2(0,T,H^2(0,1))$, $\frac{\partial e}{\partial t} \in L^2(0,T,L^2(0,1))$ and $\bar{f} \in L^2(0,T,H^1(0,\beta_2))$. Since $u_0 \in H^2(0,1)$, (34) holds and a first estimate is obtained by using the Gronwall lemma (see for instance [14, page 14]) for the quantity $\left|\left|e\right|\right|_{L^2(0,1)}^2 + \frac{\varepsilon}{2\beta_2^2}\left|s-s_h\right|^2$ in (49):

$$\left|\left|e(t)\right|\right|_{L^2(0,1)}^2 + \frac{\varepsilon}{2\beta_2^2}\left|s(t)-s_h(t)\right|^2 \leq Ch^2, \qquad \forall t \in (0,T). \tag{50}$$

By integrating (48) on $(0,T)$ and by using (34) and (50), there exists a constant C such that:

$$\left|\left|\frac{\partial e}{\partial x}\right|\right|_{L^2(U_T)}^2 \leq Ch^2.$$

This leads to the estimate (37) of the theorem by noticing that

$$\left|\left|\dot{s}-\dot{s}_h\right|\right|^2 \leq \left|\left|\frac{\partial e}{\partial x}\right|\right|_{L^2(U_T)}^2 \leq Ch^2.$$

This ends the proof of Theorem 4. □

Remark 1 By using a duality argument, it is possible to prove that (see [3]):

$$\left|\left|u-u_h\right|\right|_{L^2(U_{\tilde{T}})} \leq C\left(h^2 + \left|\left|\dot{s}-\dot{s}_h\right|\right|_{L^2(0,\tilde{T})}\right),$$

where C is a constant independent of h. This estimate shows that the L^2-error on the velocity u is governed by the H^1-error on the boundary function s.

The *a priori* error analysis is now closed. The *a posteriori* error analysis is summarized in the next theorem.

Theorem 5 (A posteriori error estimate) *Assume that* $u_0 \in H^2(0,1) \cap V$ *and* $\bar{f} \in L^2(0,T,H^1(0,\beta_2))$ *and set* $h_i = |x_{i+1} - x_i|$, $h = \max_i h_i$ *and* $J_i = (x_i, x_{i+1})$, $i = 0,\ldots,N$. *Then there exists* $0 < \tilde{T} \leq T$ *and a constant* C, *all independent of* $h > 0$, *such that, for all* t *in* $(0,\tilde{T})$:

$$||u - u_h(t)||^2_{L^2(0,1)} + \frac{\varepsilon}{2\beta_2^2} |s(t) - s_h(t)|^2 + \frac{\varepsilon}{8\beta_2^2} \int_0^t \left|\left| \frac{\partial(u - u_h)}{\partial x} \right|\right|^2 d\tau$$
$$\leq C \sum_{i=0}^{N} h_i^2 \left\{ ||F(u_h)||^2_{L^2(0,t,L^2(J_i))} + h_i^2 \, ||u_0||^2_{H^2(J_i)} \right\},$$

where

$$F(u_h) = f_h - \frac{\partial u_h}{\partial t} - \frac{\alpha}{s_h(t)} u_h \frac{\partial u_h}{\partial x} + \frac{\dot{s}_h(t)}{s_h(t)} x \frac{\partial u_h}{\partial x}. \tag{51}$$

Proof Recall that the error $u - u_h$ is denoted by e. The weak formulation (33) allows us to write for all v in V:

$$\begin{aligned}
&\left(\frac{\partial e}{\partial t}, v\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial e}{\partial x}, \frac{\partial v}{\partial x}\right) = \qquad (52)\\
&\left(\frac{\partial u}{\partial t}, v\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) - \left[\left(\frac{\partial u_h}{\partial t}, v\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right)\right] = \\
&\int_0^1 f v dx + \frac{\dot{s}(t)}{s(t)} \int_0^1 x \frac{\partial u}{\partial x} v dx - \frac{\alpha}{s(t)} \int_0^1 u \frac{\partial u}{\partial x} v dx \\
&- \left[\left(\frac{\partial u_h}{\partial t}, v\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right)\right].
\end{aligned}$$

Definition (51) together with (52) lead to:

$$\begin{aligned}
&\left(\frac{\partial e}{\partial t}, v\right) + \frac{\varepsilon}{s(t)^2}\left(\frac{\partial e}{\partial x}, \frac{\partial v}{\partial x}\right) = \qquad (53)\\
&\int_0^1 f v dx + \frac{\dot{s}(t)}{s(t)}\int_0^1 x\frac{\partial u}{\partial x} v dx - \frac{\alpha}{s(t)}\int_0^1 u\frac{\partial u}{\partial x} v dx\\
&-\left[\left(\frac{\partial u_h}{\partial t}, v\right) + \frac{\varepsilon}{s_h(t)^2}\left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right) + \left(\frac{\varepsilon}{s(t)^2} - \frac{\varepsilon}{s_h(t)^2}\right)\left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right)\right] =\\
&\int_0^1 f v dx + \frac{\dot{s}(t)}{s(t)}\int_0^1 x\frac{\partial u}{\partial x} v dx - \frac{\alpha}{s(t)}\int_0^1 u\frac{\partial u}{\partial x} v dx\\
&-\left(\frac{\varepsilon}{s(t)^2} - \frac{\varepsilon}{s_h(t)^2}\right)\left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right) - \int_0^1 f_h v dx - \frac{\dot{s}_h(t)}{s_h(t)}\int_0^1 x\frac{\partial u_h}{\partial x} v dx\\
&+\frac{\alpha}{s_h(t)}\int_0^1 u_h\frac{\partial u_h}{\partial x} v dx + \left[\int_0^1 F(u_h) v dx - \frac{\varepsilon}{s_h^2(t)}\int_0^1 \frac{\partial u_h}{\partial x}\frac{\partial v}{\partial x} dx\right],
\end{aligned}$$

where $F(u_h)$ is the *residual* quantity defined by (51). Since $\frac{\partial^2 u_h}{\partial x^2} = 0$, on J_i, $0 \le i \le N$, the last term of (53) between brackets becomes after integration by parts:

$$\sum_{i=0}^{N}\left\{\int_{x_i}^{x_{i+1}} F(u_h) v dx - \frac{\varepsilon}{s_h^2(t)}\left(\frac{\partial u_h}{\partial x}(x_{i+1})v(x_{i+1}) - \frac{\partial u_h}{\partial x}(x_i)v(x_i)\right)\right\}.$$

The solution u_h to (35) also satisfies for all v_h in V_h:

$$\begin{aligned}
&\sum_{i=0}^{N}\left\{\int_{x_i}^{x_{i+1}} F(u_h) v_h dx \right. \qquad (54)\\
&\left. \quad -\frac{\varepsilon}{s_h(t)^2}\left(\frac{\partial u_h}{\partial x}(x_{i+1})v_h(x_{i+1}) - \frac{\partial u_h}{\partial x}(x_i)v_h(x_i)\right)\right\} = 0.
\end{aligned}$$

Equation (54) is added to relationship (53) with $v = e \in V$ and $v_h = R_h e$. Since $R_h e$ is the interpolant of e, we have $(e - R_h e)(x_i) = 0$, $i = 0, \dots N+1$. By using Young inequalities, (43), (45), (47), interpolation results and (21), (36), there exist constants C_1, C_2 and C_3 such that:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,||e||^2 + \frac{\varepsilon}{2\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \\
&\leq C_1 \sum_{i=0}^{N} ||F(u_h)||_{L^2(J_i)}\, h_i\, |e|_{H^1(J_i)} + C_2\,||e||^2 + C_3\,|s-s_h|^2 \\
&\leq C_1 \left(\sum_{i=0}^{N} ||F(u_h)||^2_{L^2(J_i)}\, h_i^2\right)^{1/2} |e|_{H^1(0,1)} + C_2\,||e||^2 + C_3\,|s-s_h|^2 .
\end{aligned}
$$

The term $\frac{\varepsilon}{4\beta_2^2}\frac{d}{dt}\,|s-s_h|^2$ is added to the left and right sides of this inequality. Hence, (46) and a Young inequality imply there exist constants C_4, C_5 and C_6 such that:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||e||^2 + \frac{\varepsilon}{2\beta_2^2}\,|s-s_h|^2\right) + \frac{\varepsilon}{8\beta_2^2}\left|\left|\frac{\partial e}{\partial x}\right|\right|^2 \leq \\
&C_4 \sum_{i=0}^{N} ||F(u_h)||^2_{L^2(J_i)}\, h_i^2 + C_5\,||e||^2 + C_6\,|s-s_h|^2 .
\end{aligned}
\tag{55}
$$

By using Gronwall's lemma and by noticing that the initial conditions satisfy $||e(0)||^2_{L^2(0,1)} \leq C\sum_{i=1}^{N} h_i^4\,||u_0||^2_{H^2(J_i)}$ and $|s(0)-s_h(0)| = 0$, we obtain:

$$
\begin{aligned}
&||u-u_h(t)||^2_{L^2(0,1)} + \frac{\varepsilon}{2\beta_2^2}\,|s(t)-s_h(t)|^2 \leq \\
&C\sum_{i=1}^{N} h_i^4\,||u_0||^2_{H^2(J_i)} + C\sum_{i=0}^{N} h_i^2 \int_0^t ||F(u_h)||^2_{L^2(J_i)}\,d\tau.
\end{aligned}
$$

The estimation for $\int_0^t \left|\left|\frac{\partial(u-u_h)}{\partial x}\right|\right|^2_{L^2(0,1)} d\tau$ is then obtained by integration of (55) on $(0,t)$ and the conclusion follows. □

6 Numerical Experiments

In order to control the optimality of our error estimates, problem (1)-(6) has to be discretized in space and time. In order to do this, a splitting algorithm is chosen combined with an Arbitrary Lagrangian-Eulerian (ALE) method to discretize the advective part of (1) on the moving domain governed by (5)-(6). Naturally several others numerical methods for treating the Burgers equation exist in the literature, see for instance [10,11] for the case without a free surface.

Let us remark that the discretization in time which is chosen here is a little bit different from the one proposed in [3] for treating the two and three dimensional free surface problem. Actually, in the one-dimensional problem, the velocity u is not assumed to be divergence free, otherwise the problem (1)-(6) becomes obvious. Moreover, as outlined in [3], the volume-of-fluid method (VOF) used together with a characteristics method to discretize the advection part of the two and three dimensional free surface flow problem is not consistent if u is not divergence free. It is the reason why an ALE method is taken here.

Assume that, at time t^n, an approximation s^n of $s(t^n)$ is given together with an approximation $u_h^n(y)$ of $u(y,t^n)$ given by a piecewise polynomial function of degree 1 on the mesh y_j^n, $j=0,\ldots,N+1$ defined by $y_j^n = jh^n$ with a constant spatial step $h^n = s^n/(N+1)$ at each time t^n. Let τ^n be a time step. In order to compute s^{n+1}, u_h^{n+1} at time $t^{n+1} = t^n + \tau^n$, the following scheme is used:

1. Equation (5) is solved by an explicit Euler method, *i.e.*:
$$s^{n+1} = s^n + \tau^n u_h^n(s^n). \tag{56}$$
2. We define $h^{n+1} = s^{n+1}/(N+1)$, $y_j^{n+1} = jh^{n+1}$, $j=0,1,\ldots,N+1$ and the advection part related to (1)-(4) is solved by using an ALE upwind explicit method, *i.e.*:
$$\begin{aligned} u_j^{n+1/2} = u_j^n - \frac{\tau^n}{h^n}\big[&(\alpha u_j^n - v_j^n)^+(u_j^n - u_{j-1}^n) \\ &+(\alpha u_j^n - v_j^n)^-(u_{j+1}^n - u_j^n)\big], \end{aligned} \tag{57}$$
where $u_j^n = u_h^n(y_j^n)$, v_j^n is the velocity of the node number j, *i.e.* $v_j^n = \frac{y_j^{n+1}-y_j^n}{\tau^n}$ and the notations γ^+ and γ^- are defined by $\gamma^+ = \max(\gamma,0)$ and $\gamma^- = \min(\gamma,0)$.
3. The diffusion part of (1)-(4) is solved by using a Galerkin method on the space
$$\begin{aligned} V_h^{n+1} = \big\{&g : [0,s^{n+1}] \to \mathbb{R},\ g \text{ continuous },\ g(0)=0, \\ &g|_{[y_j^{n+1},y_{j+1}^{n+1}]} \text{ polynomial of degree } 1, j=0,\ldots,N\Big\}, \end{aligned}$$

that is: find $u_h^{n+1} \in V_h^{n+1}$ satisfying for all $v_h \in V_h^{n+1}$:

$$\int_0^{s^{n+1}} \frac{u_h^{n+1} - u_h^{n+1/2}}{\tau^n} v dy + \varepsilon \int_0^{s^{n+1}} \frac{\partial u_h^{n+1}}{\partial y} \frac{\partial v}{\partial y} dy = \int_0^{s^{n+1}} \bar{f} v dy \tag{58}$$

where $u_h^{n+1/2} \in V_h^{n+1}$ is such that $u_h^{n+1/2}(y_j^{n+1}) = u_j^{n+1/2}$, $j = 0, \ldots, N+1$.

Let us remark that, if the time step τ^n is sufficiently small with respect to $\frac{h^n}{\max_j |u_j^n|}$ in (57) (CFL condition), then our algorithm is stable. Moreover, if the time step tends to zero, then the nodes y_j^n are close to the points $y_j(t^n)$ given by $y_j(t^n) = s(t^n) x_j$ in Sect. 5.

In the following numerical results, the time step τ^n will be as small as possible in order to obtain a problem nearly continuous in time and to verify our error estimates obtained in Sect. 5.

Numerical implementation is made with MATLAB$^{\text{TM}}$. The values of the parameters are $\beta_1 = 0.01$, $\beta_2 = 2$, $T = 0.5$, $s_0 = 1$, $\alpha = 1$ and $\varepsilon = 1$. Let us consider $s(t) = 1 - t$, $t \in (0, T)$ and $u(x,t) = \sin\left(\frac{3\pi}{2} \frac{x}{s(t)}\right)$, $(x,t) \in Q_T$. Let N and M be two given positive integers. The time step is given by $\tau = T/M$ and is assumed to be constant.

Figure 1 illustrates the convergence of the approximations of the boundary function and velocity in the various norms when $N = 10, 20, 40, 80, 160$ and $\tau = 6.25 \cdot 10^{-4}$ ($M = 800$). Remark that τ is small compared to $\frac{h^n}{\max |u^n|}$.

These results are in agreement with the convergence order obtained in Sect. 5. In order to verify our *a posteriori* analysis, let us point out the effectivity index defined by:

$$I_{\text{eff}} = \frac{\left(\sum_{i=0}^N \left[h_i^2 \left|\left| f - \frac{\partial u_h}{\partial t} - \frac{\alpha}{s_h(t)} u_h \frac{\partial u_h}{\partial x} \right|\right|^2_{L^2(0,T,L^2(J_i))} + h_i^4 \, ||u_0||^2_{H^2(J_i)} \right] \right)^{1/2}}{\left|\left| \frac{\partial (u - u_h)}{\partial x} \right|\right|_{L^2(Q_T)}}.$$

In Table 1, the effectivity index I_{eff} is illustrated for $\tau = 6.25 \cdot 10^{-4}$ ($M = 800$) and when $N = 10, 20, 40, \ldots, 640$.

Since the effectivity index remains approximately constant when the spatial step tends to zero, the *a posteriori* error estimates obtained in Sect. 5 are optimal. Notice also that the effectivity index is the same when τ is smaller, for instance $\tau = 3.125 \cdot 10^{-4}$, and so does not depend on the choice of τ.

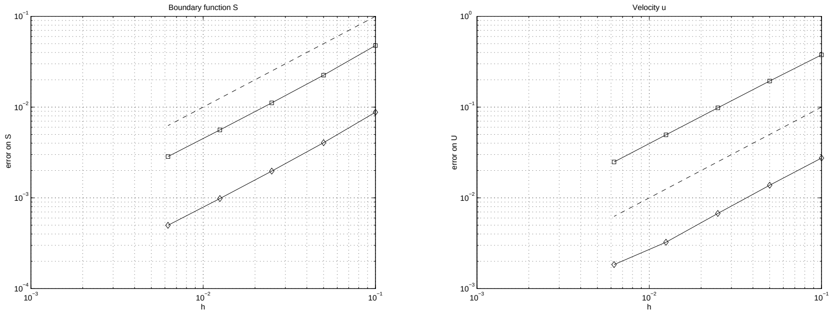

Fig. 1. Convergence orders when $\tau = 6.25 \cdot 10^{-4}$ is fixed (squares and diamonds: numerical results, dashed line: line of slope 1). Left: Log log scale plot of error on the boundary $||s - s_h||_{L^\infty(0,T)}$ (diamonds) and $||s - s_h||_{W^{1,\infty}(0,T)}$ (squares). Right: Log log scale plot of error on the velocity $||u - u_h||_{L^2(0,T,L^2(0,1))}$ (diamonds) and $||u - u_h||_{L^2(0,T,H^1(0,1))}$ (squares).

$N+1$	10	20	40	80	160	320	640
I_{eff}	3.2991	3.2631	3.2333	3.2142	3.2009	3.1876	3.1671

Table 1. Effectivity index for $\tau = 6.25 \cdot 10^{-4}$ and various N.

Finally Fig. 2 illustrates the convergence of the approximations of the boundary function and velocity in the various norms when the time step τ is divided by 2 each time that the spatial step h is divided by 2.

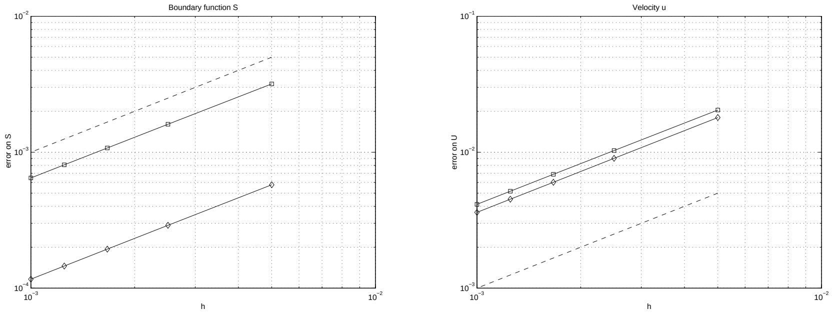

Fig. 2. Convergence orders when τ is divided by 2 each time that h is divided by 2 (squares and diamonds: numerical results, dashed line: line of slope 1). Left: Log log scale plot of error on the boundary $||s - s_h||_{L^\infty(0,T)}$ (diamonds) and $||s - s_h||_{W^{1,\infty}(0,T)}$ (squares). Right: Log log scale plot of error on the velocity $||u - u_h||_{L^2(0,T,L^2(0,1))}$ (diamonds) and $||u - u_h||_{L^2(0,T,H^1(0,1))}$ (squares).

The observed convergence orders for boundary approximation and velocity approximation are

$$||s - s_h||_{W^{1,\infty}(0,T)} + ||u||_{L^2(U_T)} + \left|\left|\frac{\partial u}{\partial x}\right|\right|_{L^2(U_T)} \leq C\,(h + \tau)$$

implying that our method is first order accurate.

7 Conclusion

The problem treated here is a simplification in one space-dimension of a free surface problem occurring in fluid mechanics. We have proved that this problem has a solution, locally in time, in some well-chosen functional spaces. The approximation by finite elements for a semi-discretization in space converges to the exact solution when the mesh size in space becomes very fine and *a priori* and *a posteriori* error estimates have been obtained. A time splitting scheme allows us to verify the theoretical results by taking a time step which is small compared to the spatial mesh size.

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