

A Two-Grids/Projection Algorithm for Obstacle Problems

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Abstract

In order to emphasize the possible relation between discontinuous and continuous approximations on different meshes, a two-grids method for the resolution of parabolic variational inequality problems is presented. The numerical methodology combines a time splitting algorithm to decouple a diffusion phenomenon from an obstacle problem. The diffusion problem is solved by using finite differences, while piecewise linear finite element techniques are used together with a Newton method for the obstacle problem. Projections are used to interpolate the solution from one grid to the other. Numerical experiments show that the resulting method has good accuracy properties.

Keywords: Two-grid method, Obstacle problem, Splitting scheme

1 Introduction

Multigrids methods are widely used in number of applications, see *e.g.* [1]. For instance, an approximation of some physical unknown is obtained on one grid and another finer mesh may permit to refine locally the solution [2] or to use different types of approximations on each of the grids [3, 4, 5].

In this paper, the coupling between a discontinuous approximation of some unknown and a continuous approximation of the same unknown is investigated. The problem of interest is an obstacle problem, formulated as a variational inequality, see *e.g.* [6]. The obstacle problem is approximated by penalty, leading

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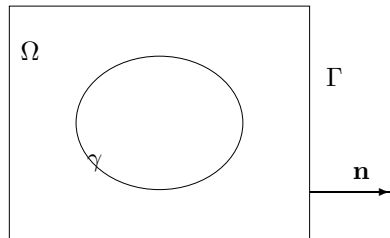


Figure 1: Geometry for the model problem: a closed line γ is included in a two-dimensional domain Ω with boundary Γ and normal vector \mathbf{n} .

to a parabolic diffusion equation with an additional nonlinear penalty term. A *Marchuk-Yanenko* time splitting scheme is used to decouple the diffusion phenomenon from the contact one, associated to the obstacle.

On the one hand, a diffusion operator may be approximated by a discontinuous approximation, for instance with finite volumes, see for instance [7, 8], or finite differences scheme, see *e.g.* [9] in the frame of computational fluid dynamics with free surfaces. On the other hand, the obstacle problem consists in constraining the value of the solution on a subdomain of the original computational domain of codimension one. In this context, the solution has to be known pointwise on the obstacle and the numerical scheme has to provide an approximation which is at least continuous. For these reasons a two-grid method is used and projection operators are defined to communicate between the two grids. Clearly the methodology discussed here also applies to distributed obstacle problems, like those encountered in [10, 11].

The structure of this paper is the following: in the next section, our model problem is described. In Sect. 3, a time splitting scheme is presented to decouple both phenomena and each step of the scheme is described. In Sect. 4, the two-grids method is described and detailed on a particular example. Finally numerical results are presented in Sect. 5 to validate the accuracy of our approach.

2 The Model Problem

The following obstacle model problem is considered. Let Ω be a bounded domain of \mathbb{R}^d , $d = 2, 3$ with boundary $\Gamma = \partial\Omega$ and let γ be a compact manifold of dimension $d - 1$ contained in Ω . Let \mathbf{n} denote the external normal vector to Ω on the boundary Γ . Notations are reported in Fig. 1 for a two-dimensional example.

The initial variational inequality is now described. Let $\psi : \gamma \rightarrow \mathbb{R}^d$ be a given continuous function. Let K_γ be the set defined by:

$$K_\gamma = \{v \in H^1(\Omega) : v \geq \psi \text{ on } \gamma\}.$$

Let $g : \Gamma \times (0, T) \rightarrow \mathbb{R}^d$ be a given function in $C^0([0, T]; H^{-1/2}(\Gamma))$. The model problem is then, for all $t > 0$, to find $u(t) \in K_\gamma$ satisfying

$$\alpha \int_\Omega \frac{\partial u}{\partial t} (v - u) dx + \chi \int_\Omega \nabla u \cdot \nabla (v - u) dx \geq \langle g, (v - u) \rangle, \quad \forall v \in K_\gamma \quad (1)$$

with given initial condition

$$u(0) = u_0 \quad (\in K_\gamma). \quad (2)$$

Here α and χ denote two given positive numbers and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Problem (1), (2) has a unique solution (see [6] for instance). Let ε be a strictly positive parameter and denote $\max(0, -v)$ by v_- . An approximation of problem (1) (2) is given by the following penalized problem: for $t > 0$, find $u_\varepsilon(t) \in H^1(\Omega)$ satisfying

$$\alpha \int_\Omega \frac{\partial u_\varepsilon}{\partial t} v dx + \chi \int_\Omega \nabla u_\varepsilon \cdot \nabla v dx - \frac{1}{\varepsilon} \int_\gamma (u_\varepsilon - \psi)_-^2 v d\gamma = \langle g, v \rangle, \quad (3)$$

for all $v \in H^1(\Omega)$ with initial condition $u_\varepsilon(0) = u_0$. Note that the solution u_ε of (3) tends to the solution u of (1) when ε tends to zero, see *e.g.* [6].

3 A Time Splitting Scheme

Let $\tau > 0$ be a given time step. Problem (3) is discretized by using an implicit Euler scheme. The initial approximation is defined by $u_\varepsilon^0 = u_0$ and $g^{n+1} := g(t^{n+1})$. For each $n \geq 0$, the problem is equivalent to find $u_\varepsilon^{n+1} \in H^1(\Omega)$ satisfying

$$\begin{aligned} \alpha \int_\Omega \frac{u_\varepsilon^{n+1} - u_\varepsilon^n}{\tau} v dx + \chi \int_\Omega \nabla u_\varepsilon^{n+1} \cdot \nabla v dx \\ - \frac{1}{\varepsilon} \int_\gamma (u_\varepsilon^{n+1} - \psi)_-^2 v d\gamma = \langle g^{n+1}, v \rangle, \quad \forall v \in H^1(\Omega). \end{aligned} \quad (4)$$

This problem is treated with the so-called *Marchuk-Yanenko* time splitting scheme, see [12]. This implies that, at each time step, the two following problems have to be solved successively:

1. Find $u_\varepsilon^{n+1/2} \in H^1(\Omega)$ satisfying:

$$\alpha \int_\Omega \frac{u_\varepsilon^{n+1/2} - u_\varepsilon^n}{\tau} v dx + \chi \int_\Omega \nabla u_\varepsilon^{n+1/2} \cdot \nabla v dx = \langle g^{n+1}, v \rangle, \quad \forall v \in H^1(\Omega). \quad (5)$$

Note that (5) is the weak formulation of the problem consisting in finding u_ε satisfying:

$$\begin{aligned} \alpha \frac{\partial u_\varepsilon}{\partial t} - \chi \Delta u_\varepsilon &= 0, & \text{in } \Omega \times (t^n, t^{n+1}), \\ \chi \frac{\partial u_\varepsilon}{\partial \mathbf{n}} &= g, & \text{on } \Gamma \times (t^n, t^{n+1}), \\ u_\varepsilon(t^n) &= u^n & \text{in } \Omega, \end{aligned}$$

after it has been discretized by one step of the backward Euler scheme.

2. Find $u_\varepsilon^{n+1} \in H^1(\Omega)$ satisfying:

$$\alpha \int_{\Omega} \frac{u_\varepsilon^{n+1} - u_\varepsilon^{n+1/2}}{\tau} v dx - \frac{1}{\varepsilon} \int_{\gamma} (u_\varepsilon^{n+1} - \psi)_-^2 v d\gamma = 0, \quad \forall v \in H^1(\Omega). \quad (6)$$

For a fixed parameter ε , the above time splitting scheme is an order one scheme in the sense that it introduces an additional error of order $\mathcal{O}(\tau^2)$ at each time step, see *e.g.* [12].

4 A Two Grids Method with Projections

The time splitting scheme detailed in Sect. 3 introduces two problems of different nature. Problem (5) is a classical time-discrete parabolic equation, while (6) is an obstacle problem involving the values of the solution on a subset of the computational domain which is of codimension one by assumption. Then finite differences schemes or finite volumes schemes are well-adapted to the resolution of the problem (5) and lead to an approximation of the solution which is piecewise constant on each cell of a structured grid. On the other hand, discontinuous approximations are not well-adapted to the obstacle problem (6) since the penalty term has to be considered in a pointwise sense. A continuous approximation is then preferable and a triangulation of the domain and piecewise linear finite element approximations are used.

The two-dimensional case is considered in the sequel. The two grids method combines a regular grid of square cells and a nested finite element structured triangulation, as shown in Fig. 2 in the case of a square domain Ω .

Let a, b be two given real numbers with $a < b$ and let us assume for simplicity that $\Omega = (a, b) \times (a, b)$. Let $I \in \mathbb{N}_+$ be a given positive integer and set $h = \frac{b-a}{I}$. Let x_{ij} be the center of the cell (i, j) denoted by K_{ij} , *i.e.* the point with coordinates given by $(a + (i - 1/2)h, a + (j - 1/2)h)$, $i, j = 1, \dots, I$.

Let u_D^n be the piecewise constant approximation of $u(x, t^n)$ with constant value u_{ij}^n on the cell K_{ij} . Problem (5) is solved with an implicit centered finite differences scheme. At each time step, $n \geq 0$, it consists in finding $u_{ij}^{n+1/2}$, $i, j = 1, \dots, I$ satisfying:

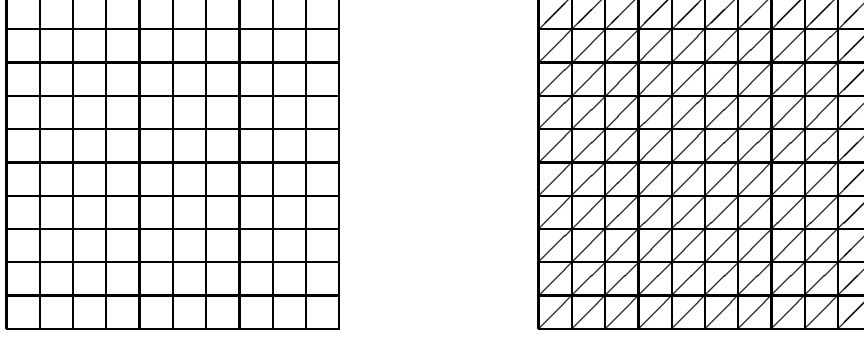


Figure 2: A two-grids method: left: structured grid of squares, right: triangulation.

$$\frac{u_{ij}^{n+1/2} - u_{ij}^n}{\tau} + \chi \frac{4u_{ij}^{n+1/2} - u_{i+1j}^{n+1/2} - u_{i-1j}^{n+1/2} - u_{ij+1}^{n+1/2} - u_{ij-1}^{n+1/2}}{h^2} = 0$$

for $i, j = 2, \dots, I-1$ and u_{ij}^n is given. Near the boundary, the finite differences scheme should be modified in order to take into account the boundary condition (6). It corresponds to solving the linear system $(I + \tau\chi A)\bar{u}^{n+1/2} = \bar{u}^n + \tau\bar{g}^{n+1}$, where \bar{u}^n is the vector with components u_{ij}^n and \bar{g}^{n+1} is the vector which represents the influence of boundary conditions.

Once the constant values $u_{ij}^{n+1/2}$ of $u_D^{n+1/2}$ on each cell K_{ij} are computed, the approximation $u_D^{n+1/2}$ is transposed on the grid points of the finite element mesh in order to solve the obstacle problem (6). Let us denote by u_h^n the approximation of u^n , which is piecewise linear on each triangle of the finite element mesh. In the following, a projection method is described to compute $u_h^{n+1/2}$ from $u_D^{n+1/2}$.

Let \mathcal{T}_h be the finite element triangulation illustrated in Fig. 2, with vertices denoted by P_{ij} , $i, j = 0, \dots, I$. Let φ_{ij} , $i, j = 0, \dots, I$, denote the basis functions of the piecewise linear finite element space based on \mathcal{T}_h . The approximation $u_h^{n+1/2}$ is computed by:

$$\int_{\Omega} u_h^{n+1/2} \varphi_{ij} dx = \int_{\Omega} u_D^{n+1/2} \varphi_{ij} dx, \quad i, j = 0, \dots, I. \quad (7)$$

Let $X_h^r(\Omega)$, $r > 0$ be the space of continuous functions which are piecewise polynomials of degree r on each element K of \mathcal{T}_h and let $X_h^0(\Omega)$ be the space of piecewise constant function on the grid of cells. Let $P_1 : X_h^0(\Omega) \rightarrow X_h^1(\Omega)$ be the projection operator defined by (7) which associates $u_h^{n+1/2}$ to $u_D^{n+1/2}$. In the case of Fig. 2, let us consider an interior point P_{ij} of \mathcal{T}_h and denote by

$K_{ij}, K_{i+1j}, K_{ij+1}$ and K_{i+1j+1} the cells adjacent to P_{ij} , see Fig. 3 (left). It can be easily shown that:

$$u_h^{n+1/2}(P_{ij}) = \frac{1}{6} \left(2u_{ij}^{n+1/2} + 2u_{i+1j+1}^{n+1/2} + u_{i+1j}^{n+1/2} + u_{ij+1}^{n+1/2} \right), \quad (8)$$

for each $P_{ij} \notin \partial\Omega$. On the boundary, similar expressions may be obtained to describe $u_h^{n+1/2}(P_{ij})$ as a weighted sum of the values $u_{ij}^{n+1/2}$.

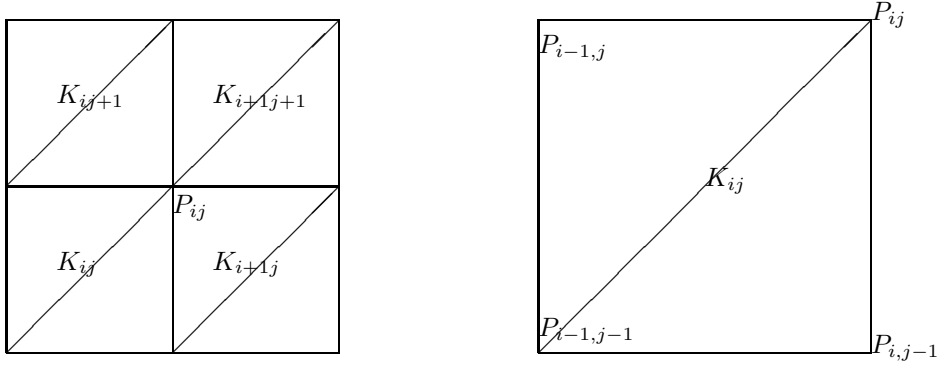


Figure 3: Internal numbering for the two-grids method. Left: numbering for the four cells adjacent to P_{ij} , right: numbering for the four grid points adjacent to a cell K_{ij} .

In the general case, both meshes are not necessarily nested and one mesh may be coarser than the other. Relationship (7) may be expressed by:

$$u_h^{n+1/2}(P_{ij}) \frac{|\Omega_{ij}|}{2} = \sum_{\substack{k,l=1,\dots,I \\ K_{kl} \cap \Omega_{ij} \neq \emptyset}} u_{kl}^{n+1/2} |K_{kl} \cap \Omega_{ij}|,$$

where Ω_{ij} is the set of all elements of the finite element mesh adjacent to the vertex P_{ij} and $|\Omega_{ij}|$ denotes the area of the set Ω_{ij} . A slight modification of this projection may be encountered for instance in [4] when the unstructured mesh is coarser than the structured one.

Let V_h denote the space of the continuous, piecewise linear functions, which are linear on each triangle $K \in \mathcal{T}_h$. Once the values of $u_h^{n+1/2}$ are obtained on the finite element vertices P_{ij} , (6) can be solved with piecewise linear continuous finite elements, namely find $u_h^{n+1} \in V_h$ satisfying:

$$\alpha \int_{\Omega} \frac{u_h^{n+1} - u_h^{n+1/2}}{\tau} v dx - \frac{1}{\varepsilon} \int_{\gamma} (u_h^{n+1} - \psi)_-^2 v d\gamma = 0, \quad \forall v \in V_h. \quad (9)$$

This implicit problem is well-posed but strongly nonlinear. It is then solved with a Newton method, as in [10]. Problem (9) is equivalent to finding u_h^{n+1} satisfying:

$$\alpha \int_{\Omega} u_h^{n+1} v dx - \frac{\tau}{\epsilon} \int_{\gamma} (u_h^{n+1} - \psi)_-^2 v d\gamma - \alpha \int_{\Omega} u_h^{n+1/2} v dx = 0.$$

Let us denote by $u_{(k)}$ the k^{th} iterate of the Newton method for a generic time step n . Set $u_{(0)} = u_h^{n+1/2}$ and $\bar{u}_k = u_{(k)} - u_{(k+1)}$. The Newton method consists, at each iteration, in finding \bar{u}_k solving the following linearized problem:

$$\begin{aligned} \int_{\Omega} \bar{u}_k v dx + \frac{2\tau}{\alpha\epsilon} \int_{\gamma} (u_{(k)} - \psi)_- \bar{u}_k v d\gamma &= \int_{\Omega} u_{(k)} v dx - \frac{\tau}{\alpha\epsilon} \int_{\gamma} (u_{(k)} - \psi)_-^2 v d\gamma \\ &\quad - \int_{\Omega} u_h^{n+1/2} v dx, \quad \forall v \in V_h. \end{aligned} \quad (10)$$

and by setting $u_{(k+1)} = u_{(k)} - \bar{u}_k$. The mass terms in (10) are easily computed with mass lumping techniques. Let us turn now to the treatment of the penalty terms. Let $M \in \mathbb{N}_+$ be a given positive integer and let Q_1, Q_2, \dots, Q_M be M points lying on γ , called *control points*. Then each integral on γ may be approximated by the quadrature formula:

$$\int_{\gamma} f(s) d\gamma \simeq \sum_{i=1}^M \frac{1}{2} [f(Q_i) + f(Q_{i+1})] \|Q_{i+1} - Q_i\|, ,$$

with the convention $Q_{M+1} := Q_1$ when γ is a closed curve. With these remarks, a Newton method may be implemented to solve iteratively the problem (10) until convergence is achieved.

Finally the solution u_h^{n+1} is projected back on the grid of square cells. Let ψ_{ij} , $i, j = 1, \dots, I$ be the basis functions of $X_h^0(\Omega)$ based on the grid of square cells. Let $P_2 : X_h^1(\Omega) \rightarrow X_h^0(\Omega)$ be the projection operator which associates the piecewise constant function u_D^{n+1} to u_h^{n+1} and which is implicitly defined by:

$$\int_{\Omega} u_D^{n+1} \psi_{ij} dx = \int_{\Omega} u_h^{n+1} \psi_{ij} dx. \quad (11)$$

This operator can be explicitly computed. For all $i, j = 1, \dots, I$ let $(i-1, j-1)$, $(i-1, j)$, (i, j) and $(i, j-1)$ the indices of the grid points of the finite element mesh which are the corners of K_{ij} , as illustrated in Fig. 3 (right). Relationship (11) gives:

$$u_{ij}^{n+1} = \frac{1}{6} [2u_h^{n+1}(P_{i-1,j-1}) + 2u_h^{n+1}(P_{ij}) + u_h^{n+1}(P_{i-1,j}) + u_h^{n+1}(P_{i,j-1})].$$

In the general case when the finite element mesh and the structured grid of cells are not nested and/or if one mesh is coarser than the other, (11) leads to:

$$u_{ij}^{n+1}h^2 = \sum_{\substack{K \in \mathcal{T}_h \\ K \cap K_{ij} \neq \emptyset}} \frac{|K \cap K_{ij}|}{3} \sum_{P_{kl} \in K} u_h^{n+1}(P_{kl}).$$

From the implementation point of view the diffusion problem (5) leads to an symmetric linear system which is solved classically with a conjugate gradient method and an incomplete Cholesky preconditioner. Problems (10) lead to the solution of symmetric linear systems, with the same properties as the ones encountered in [10]. In particular, the condition number of these systems is proportional to $\frac{1}{\sqrt{\varepsilon}}$. A conjugate gradient algorithm with incomplete Cholesky preconditioner is also used to solve each Newton step.

5 Numerical Results

Numerical results are presented to validate our methodology. The special two-dimensional case $\Omega = (-1, +1) \times (-1, +1)$ is considered. The obstacle curve γ is the circle defined by $\gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = (0.7)^2\}$. Let $M = 500$ be the number of control points on γ , the control points Q_k , $k = 1, \dots, M$ being defined by $Q_k = (0.7 \sin(\frac{2\pi}{M}k), 0.7 \cos(\frac{2\pi}{M}k))$.

Let the function ψ be identically zero on γ and $g : \Gamma \rightarrow \mathbb{R}$ be given by:

$$g(x, y, t) = \begin{cases} -10 \sin(6\pi t), & \text{if } x = -1, \\ 10 \sin(6\pi t), & \text{if } x = +1, \\ 0, & \text{otherwise.} \end{cases}$$

The initial condition is $u_0 \equiv 0$.

The physical parameters are $\alpha = 1$, $\chi = 1$ and the time step is $\tau = 0.05$. The diffusion step is solved with a preconditioned conjugate gradient algorithm with a stopping criterion of 10^{-3} on the residual. The linear system appearing in the Newton method are solved with the same conjugate gradient algorithm with a stopping criterion of 10^{-6} on the residual. The Newton method is assumed to converge if the relative difference between two consecutive iterates is less than 10^{-4} . It is clear from [10] that the Newton and conjugate gradient solvers have fast convergence properties. The CPU time used for the projections of the numerical solution from one mesh to the other is negligible with respect to the resolution of the linear systems.

First the case $N = 40$ and $M = 1000$ is considered and illustrated on Fig. 4 for a parameter value of $\varepsilon = 10^{-9}$. The final time is $T = 0.5$ and the time step is 0.025. Note that the solution is oscillating slightly on the curve γ with amplitude never greater than 10^{-2} .

Let us turn to the convergence properties of the algorithm. The solution has to satisfy the obstacle condition on γ . Let $N = 40$ and $M = 500$ be fixed. Let the final time be $T = 0.1$ and the time step $\tau = 1/60$. To illustrate the influence of the penalty parameter, Table 1 shows the value of $\|(u_\varepsilon - \psi)_-\|_{L^2(\gamma)}$ at time $T = 0.1$ for different values of the parameter ε , as well as the mean number of

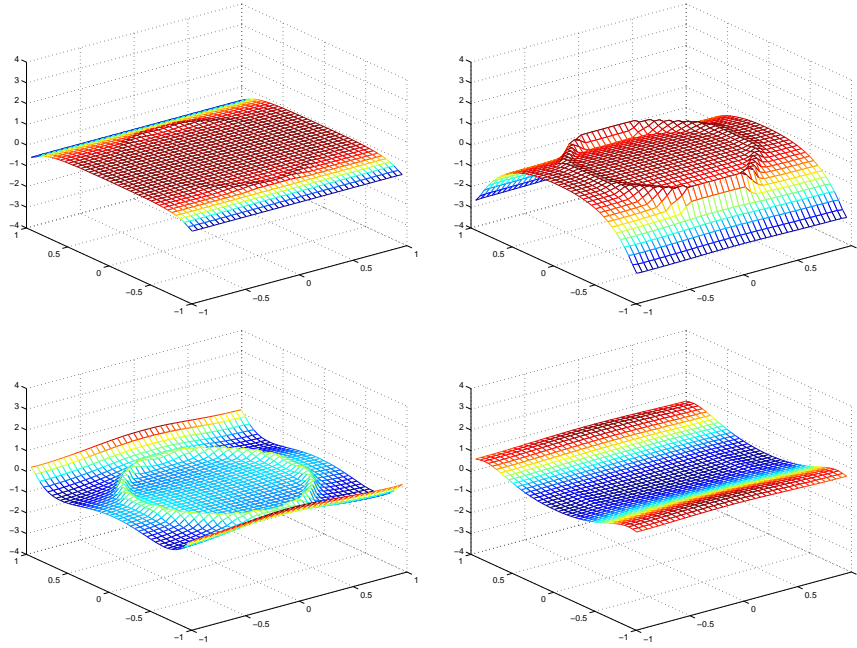


Figure 4: Numerical solution at times $t = 0.05, 0.1, 0.15$ and 0.2 (left to right, top to bottom).

iterations required for the resolution of the obstacle problem at the last time step (*i.e.* when the constraint due to the obstacle is the strongest).

ε	$\ (u_\varepsilon)_-\ _{L^2(\gamma)}$	# iter. CG	# iter. Newton
10^{-3}	0.1661	3	5
10^{-6}	0.0641	8	11
10^{-9}	0.0207	12	19
10^{-12}	0.0024	20	24

Table 1: Error on the control curve γ : behavior of the L^2 -error versus the penalty parameter ε , and number of iterates for the conjugate gradient algorithm for the resolution of the obstacle problem and the Newton method.

Table 1 illustrates the good convergence properties of the conjugate gradient/Newton method. The resolution of the diffusion problem is not discussed here. Figure 5 illustrates the behavior of the error in function of the penalty parameter. The convergence is achieved with an approximation of order $\mathcal{O}(\varepsilon^{0.3})$.

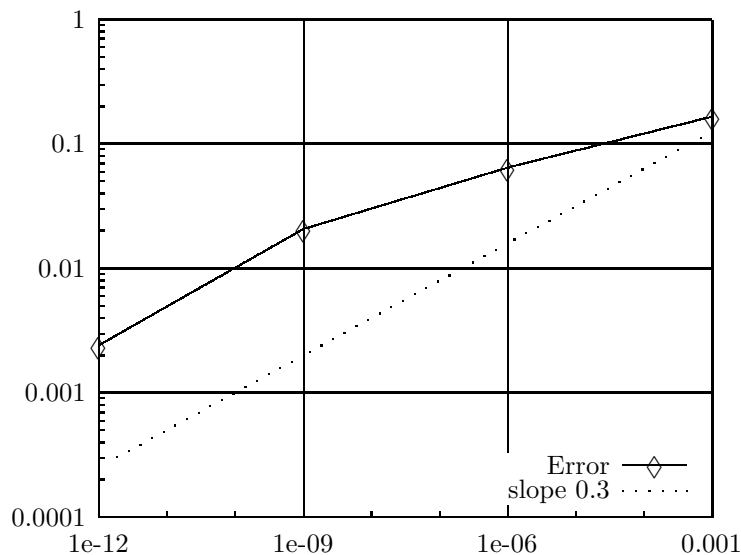


Figure 5: Log-log scale plot of the L^2 -norm of the error on the obstacle γ in function of the penalty parameter ε .

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