A Finite Element-Characteristics Method for Free Surface Flows with Bubbles

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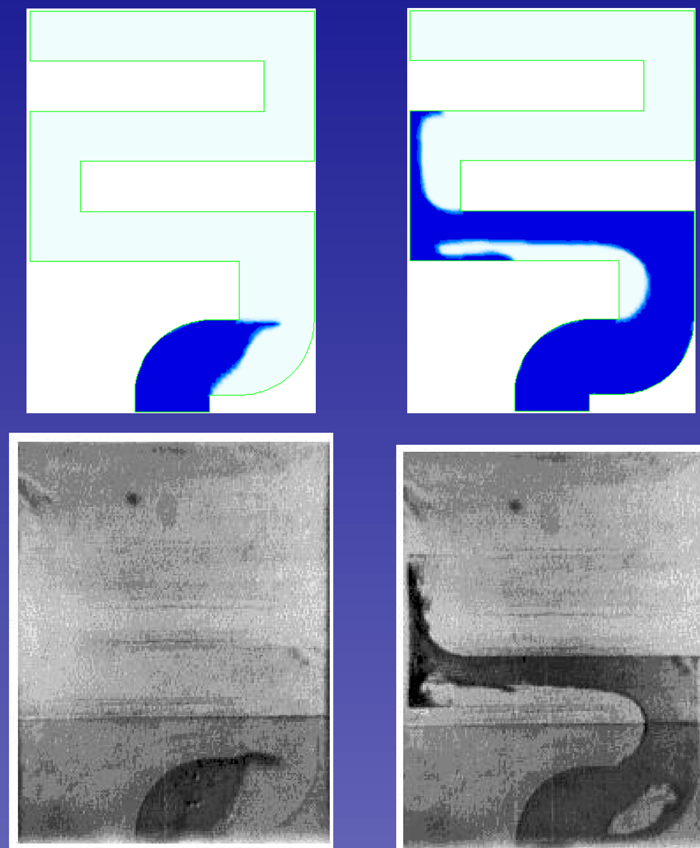
Complex 2D Flows

- Mould filling - Injection filling



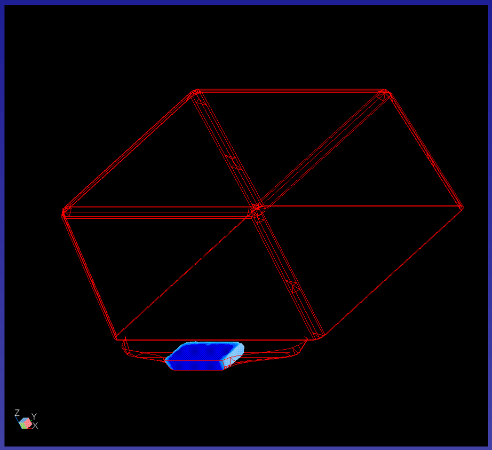
Computation

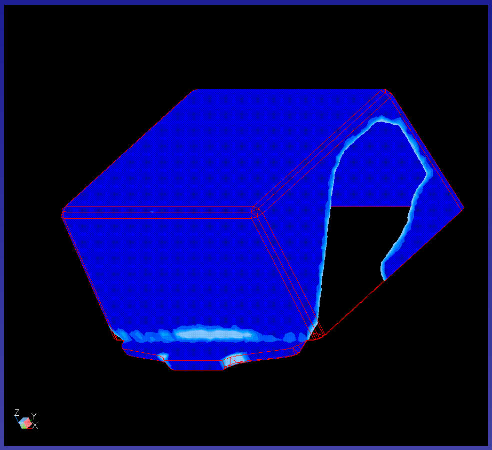
Experiment

- V. Maronnier, M. Picasso, J. Rappaz, J. Comp. Phys., 1999.

Complex 3D Flows

- Filling of a box

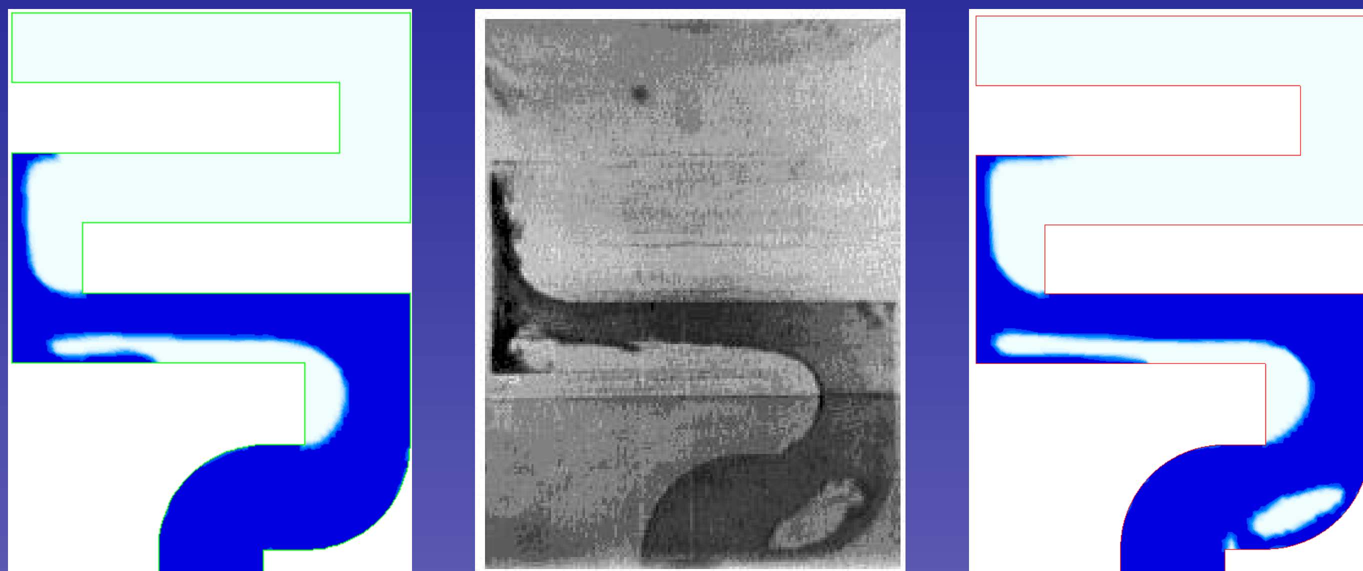




- V. Maronnier, M. Picasso, J. Rappaz, Int. J. Numer. Methods Fluids, 2003.

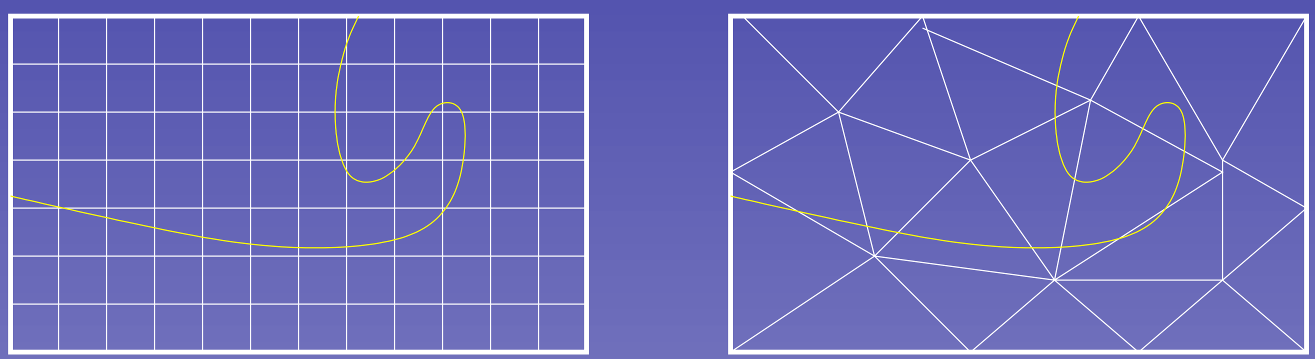
Goal

- Take into account the compressibility effect of the surrounding gas.

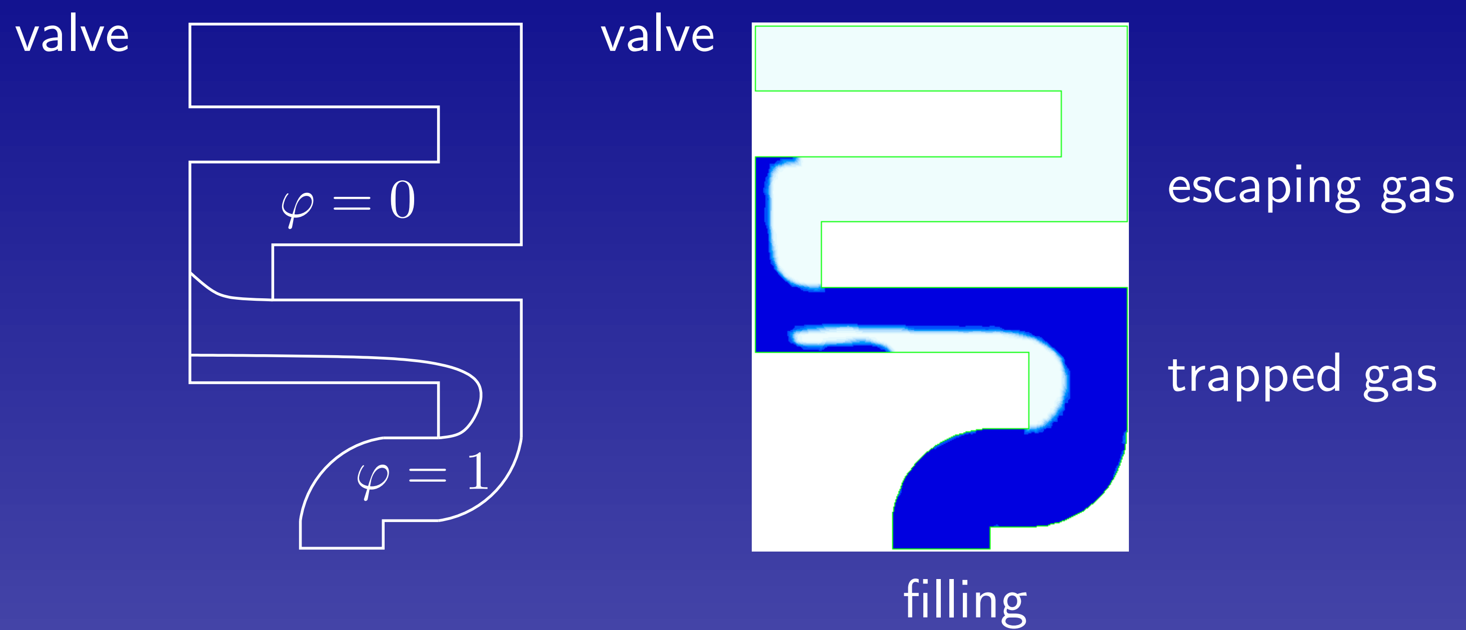


Features

- Incompressible Fluid Flow, surrounded by compressible gas.
- Volume of Fluid (VOF) formulation (Eulerian, no remeshing).
- Time splitting algorithm (implicit) to decouple advection, diffusion and gas treatment.
- Two grids method
  - Regular stuctured grid of small cubic cells (advection).
  - Finite element unstructured mesh in tetrahedrons (diffusion and gas treatment).



The Model : Volume of Fluid (VOF)



Unknowns :

- Volume fraction of liquid φ in the cavity.
- Velocity $\mathbf{u}$ and pressure p in the liquid (incompressible flow).
- Number and position of the connected components of gas.
- Pressure P in each bubble (velocity is disregarded).

Governing Equations

- **Equation for φ :** the fluid particles move with the fluid along the characteristics $(\dot{\mathbf{X}} = \mathbf{u})$

$$\frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi = 0 \ .$$

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- **Incompressible Navier-Stokes Equations in the fluid domain :**

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - 2\mathrm{div}\left(\mu \mathbf{D}(\mathbf{u})\right) + \nabla p = \rho \mathbf{g} \ ,$$

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- **Law of Ideal Gases in the gas domain (bubbles) :**

$$P \cdot V = \text{ constant in each bubble of gas } .$$

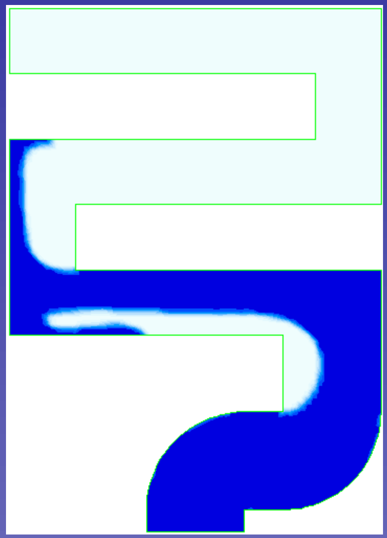
Fluid - Gas Interaction

- On the free surface (liquid-gas), force induced by compressibility of the gas :

$$-p\mathbf{n} + 2\mu D(\mathbf{u})\mathbf{n} = -(P - P_{\mathrm{atmo}})\mathbf{n}.$$

valve

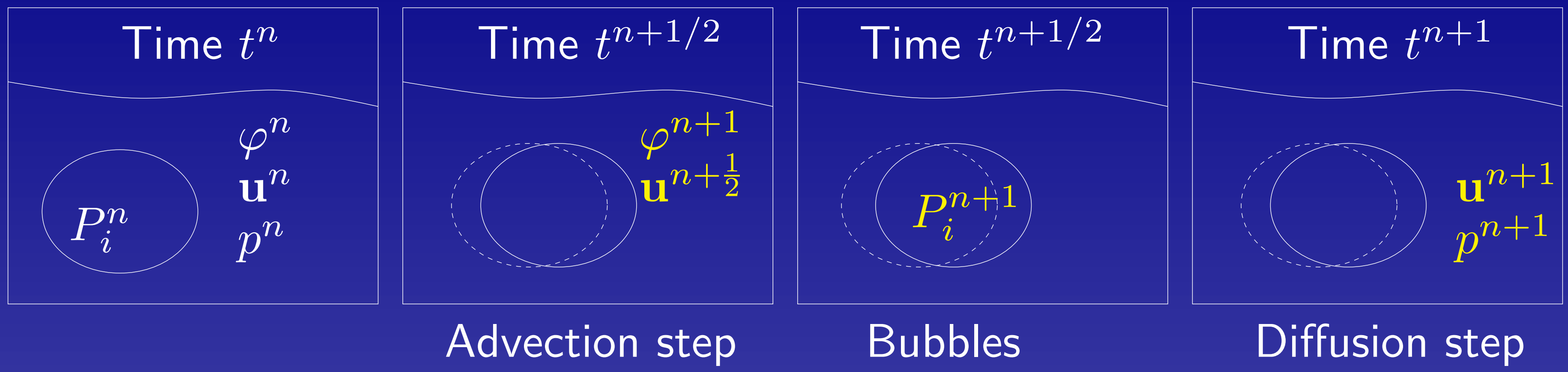
escaping gas
$\Rightarrow$ zero force



trapped gas will apply a force on the liquid-gas interface

filling

Time Discretization : A Splitting Algorithm



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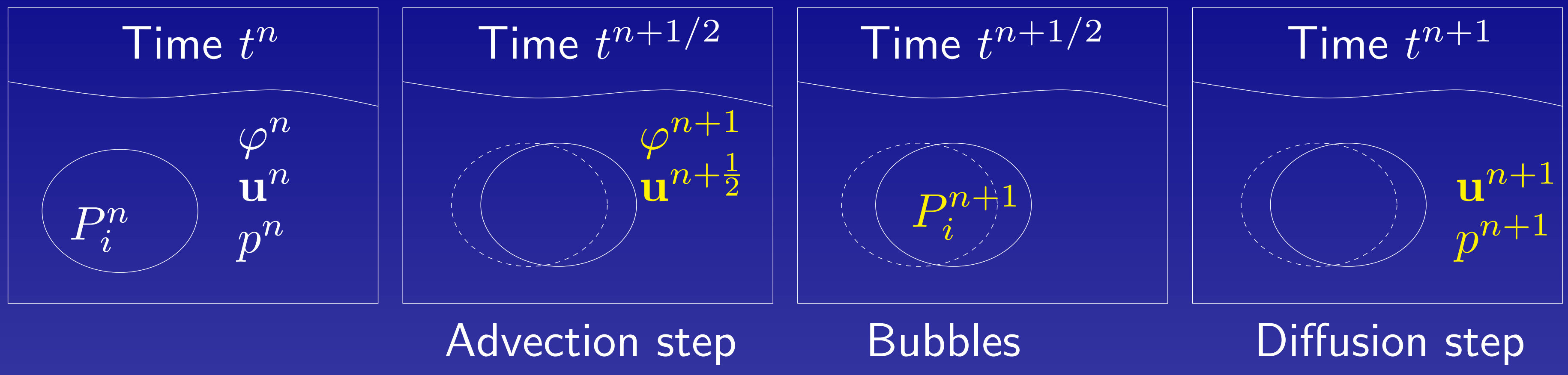

(1) Advection step

$\Rightarrow$ prediction $\mathbf{u}^{n+1/2}$ of the velocity.

$\Rightarrow$ Volume fraction of liquid φ^{n+1} and new fluid domain Ω^{n+1}.

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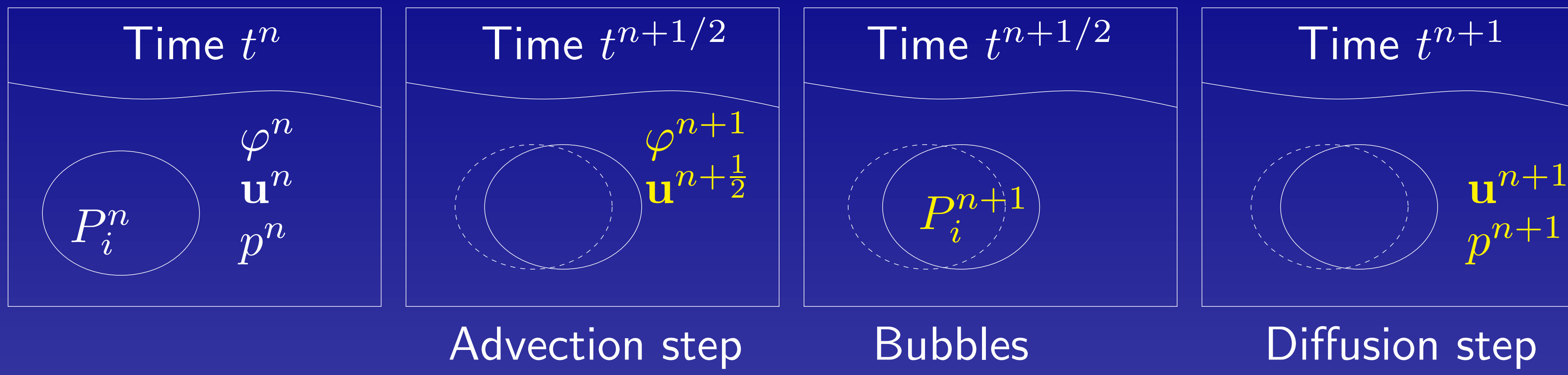

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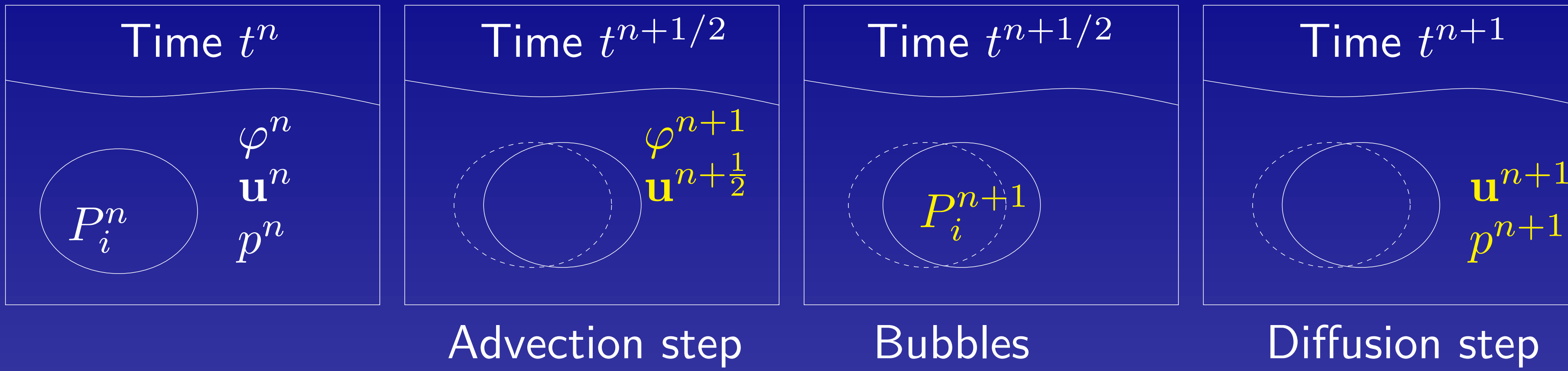

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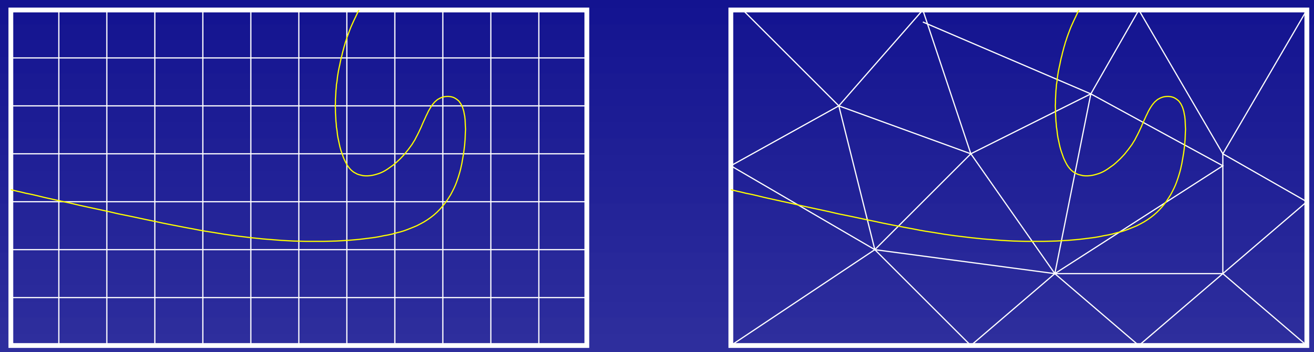
(3) Diffusion step

$\Rightarrow$ velocity $\mathbf{u}^{n+1}$ (correction) and pressure p^{n+1} in the fluid domain.

Space Discretization : Cells and Finite Elements



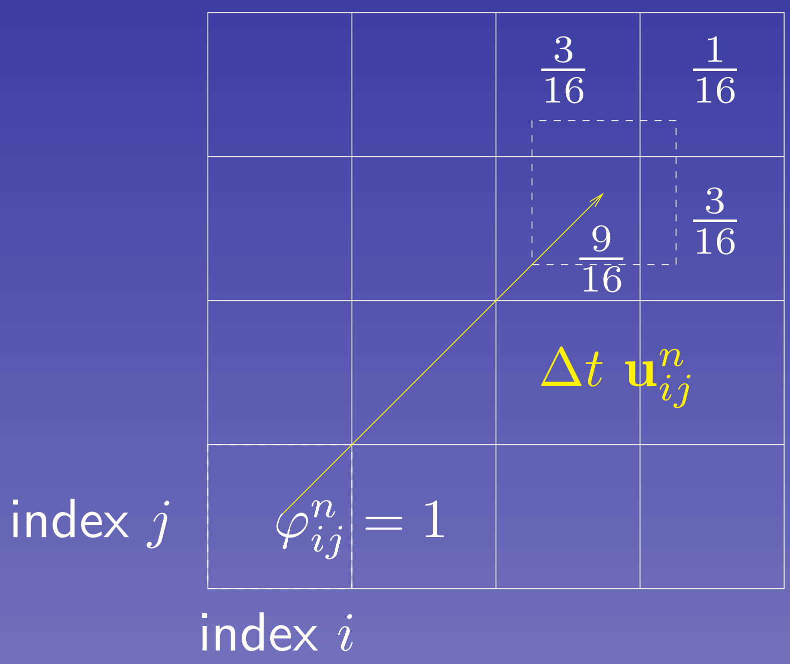
- **Advection step :**
  ⇒ forward characteristics method
  ⇒ regular grid of cubic cells

- **Bubbles and diffusion (Stokes) steps :**
  ⇒ continuous piecewise linear (stabilized) finite elements
  ⇒ unstructured finite element mesh

Advection step

- Solve the two advection problems between t^n and t^{n+1} :

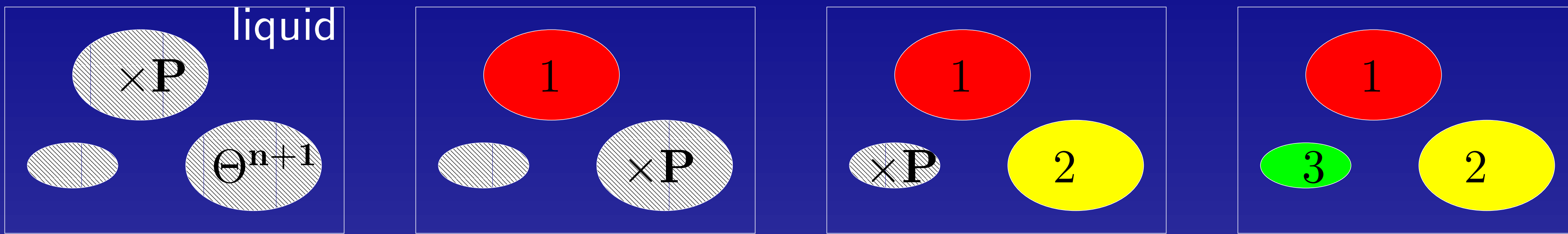
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0,$$
$$\frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi = 0.$$



- Burgers' equation : forward characteristics method with re-projection

- Numerical diffusion (SLIC, Chorin, 1980)
- Numerical compression (post-processing)

Bubbles Step : Numbering

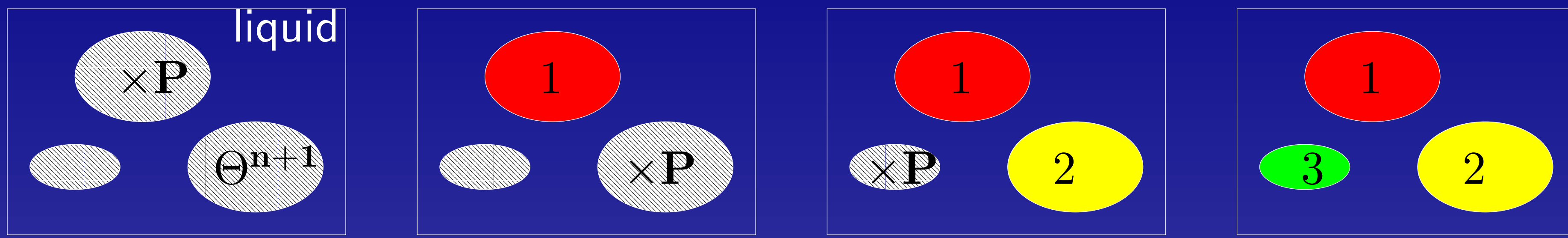


- Choose a point P in the gas domain Θ^{n+1}.
- Solve,

$$\begin{cases} -\Delta u = \delta_P, & \text{in } \Theta^{n+1} \\ u = 0, & \text{on } \partial\Theta^{n+1} \end{cases}$$

- if $u(x) \neq 0$ then $x \in$ connected component number 1 .

Bubbles Step : Numbering



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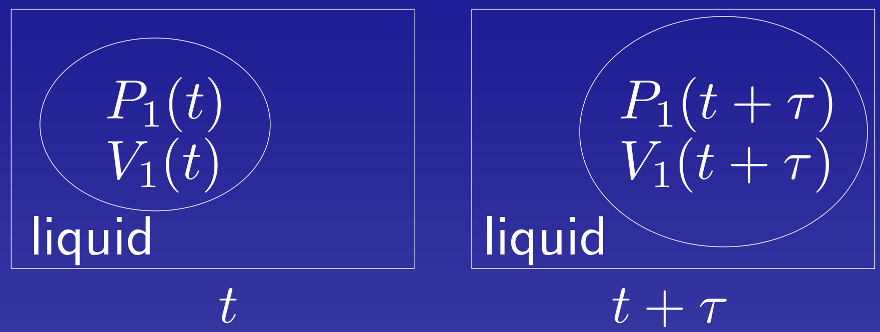
- if $u(x) \neq 0$ then $x \in$ connected component number 1 .
- Increment bubble number, change domain Θ^{n+1} and repeat until all the bubbles are numbered...
- Solve Poisson problems with continuous, piecewise linear finite elements

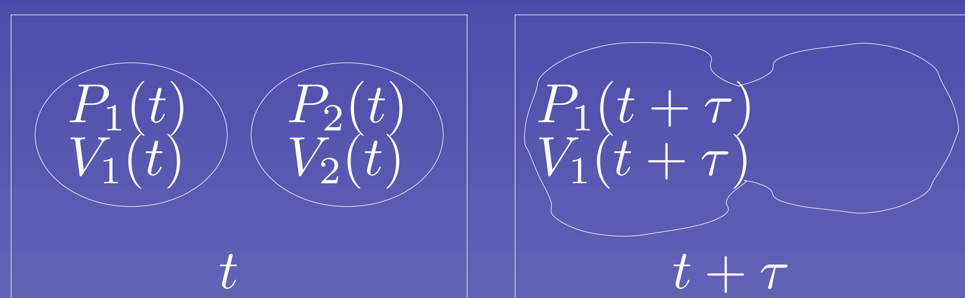
Bubbles Step : Computation of the Pressure

- Evolution of single bubble



$$P_1(t+\tau)V_1(t+\tau) = P_1(t)V_1(t) \quad .$$

- Splitting and/or collapsing of bubbles



$$P_1(t+\tau)V_1(t+\tau) = P_1(t)V_1(t) + P_2(t)V_2(t) \quad .$$

Diffusion Step

- Diffusion step (velocity correction) : solve Stokes problem in Ω^{n+1}

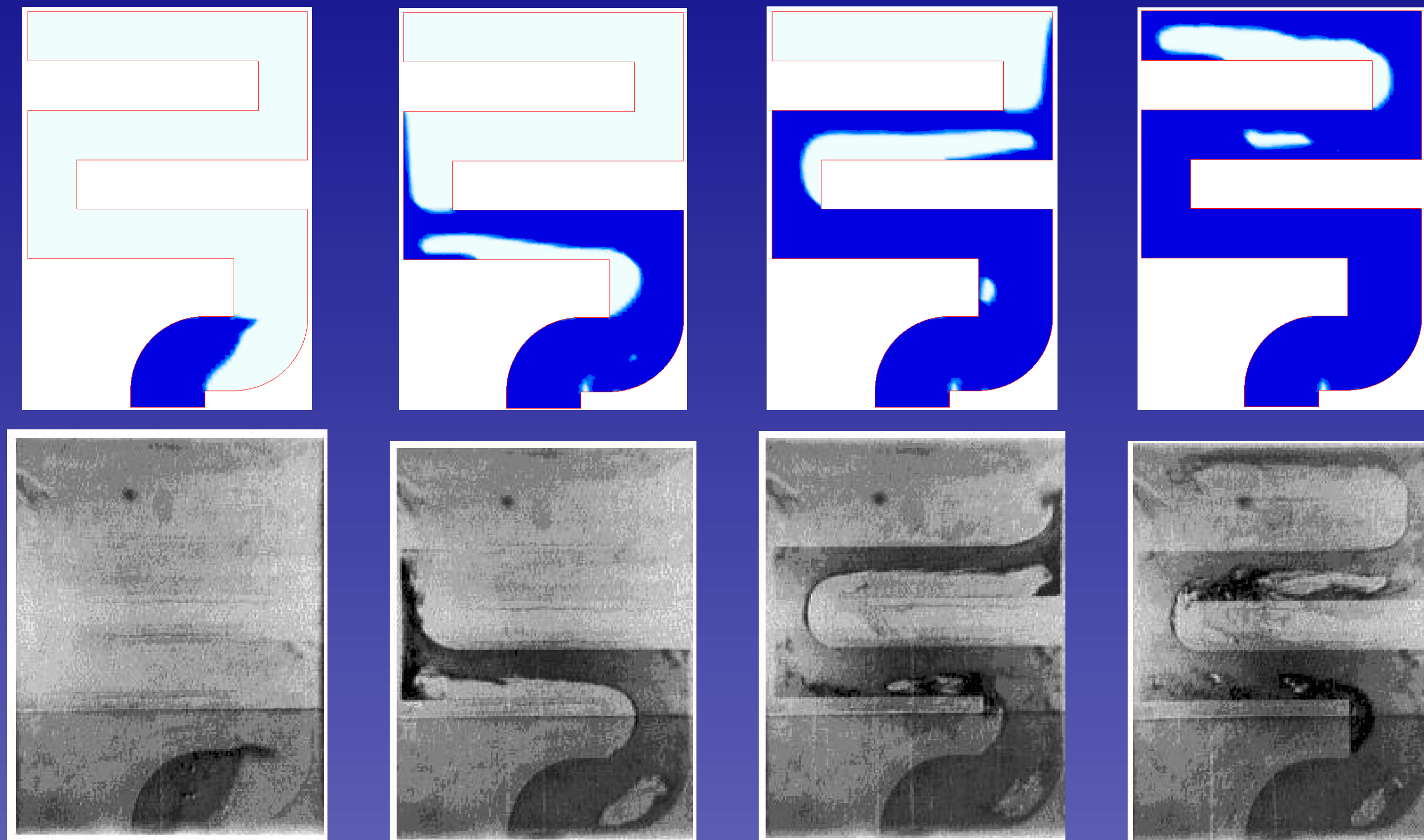
$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+\frac{1}{2}}}{\Delta t} - 2\mathrm{div}\left(\mu \mathbf{D}(\mathbf{u}^{n+1})\right) + \nabla p^{n+1} = \rho \mathbf{g},$$
$$\mathrm{div}\ \mathbf{u}^{n+1} = 0.$$

- with boundary conditions, on the free surface :

$$-p^{n+1}\mathbf{n} + 2\mu D(\mathbf{u}^{n+1})\mathbf{n} = -(P^{n+1} - P_{\mathrm{atmo}})\mathbf{n}.$$

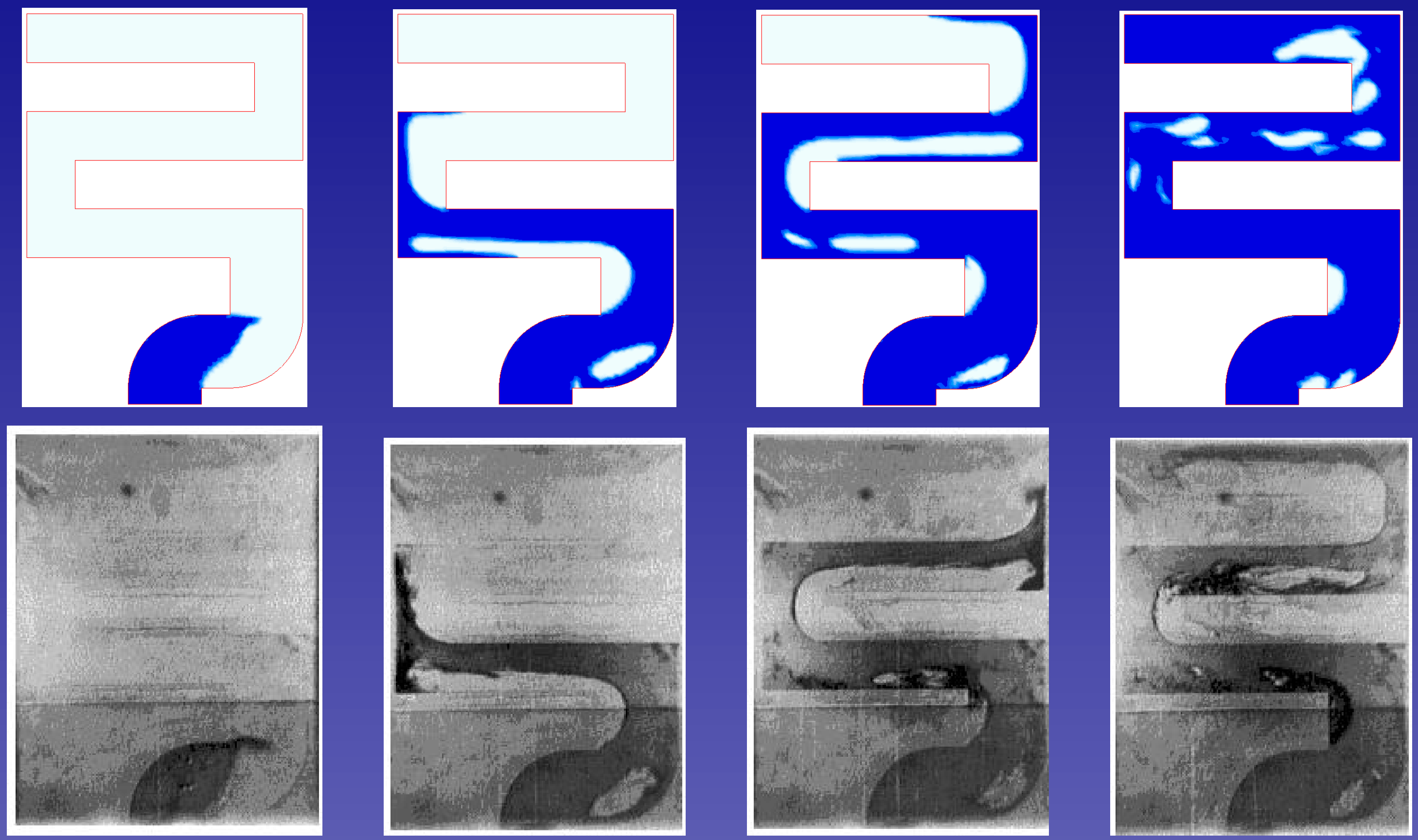
- Solve Stokes problem in Ω^{n+1} with continuous, piecewise linear stabilized finite elements (GLS, Franca-Frey, 1992)

2D-3D results : S shaped channel (without bubbles of gas)



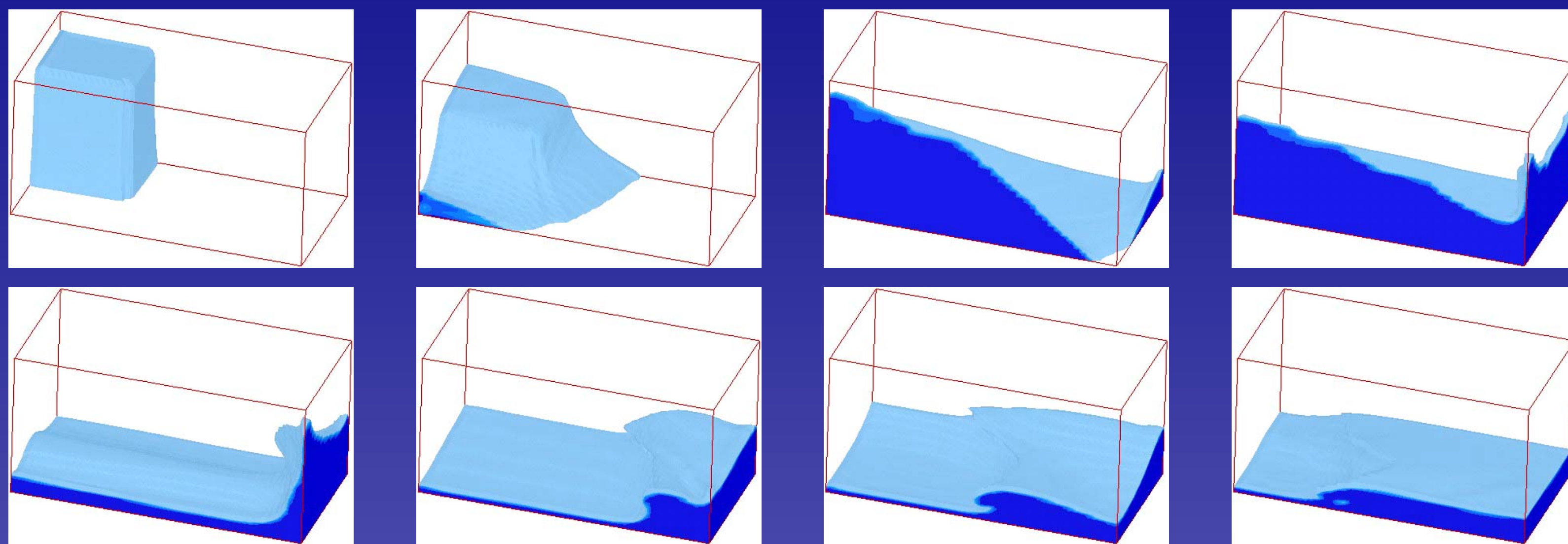
- 3D computation. 21'000 nodes and 100'000 elements for FE mesh. 1'500'000 cells. CPU $\sim$ 350 mn on a PC.

2D-3D results : S shaped channel (with bubbles of gas)



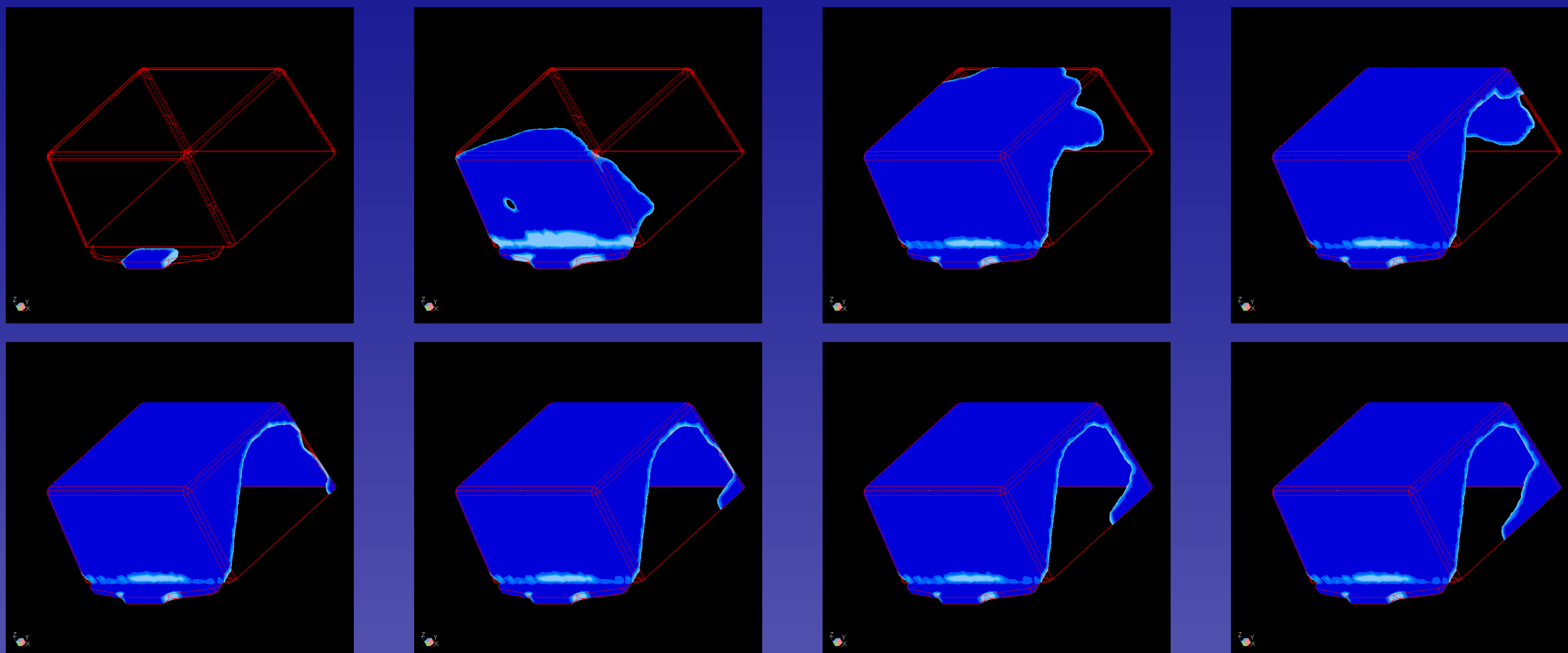
- Persistance of the bubbles of gas.
- Bubbles computation ~ 10 percent of total CPU time.

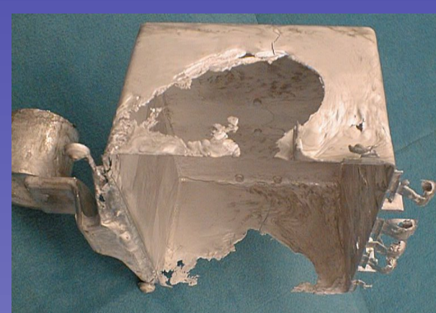
3D results : broken dam



- Regular mesh of 168750 elements. 3515625 cells. CPU 277 mn.

3D results : filling of a box





Theoretical Justification

- Model Problem in 1D

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \alpha u \dfrac{\partial u}{\partial x} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} = f \ , & x \in (0, s(t)), \quad t \in (0,T) \ , \\
u(0) = u_0 \ , & x \in (0, s_0) \ , \\
u(0,t) = 0 \ , & t \in (0,T) \ , \\
\dfrac{\partial u}{\partial x}(s(t),t) = 0 \ , & t \in (0,T) \ , \\
\dot{s}(t) = u(s(t),t) \ , & t \in (0,T) \\
s(0) = s_0 \ .
\end{cases}
$$

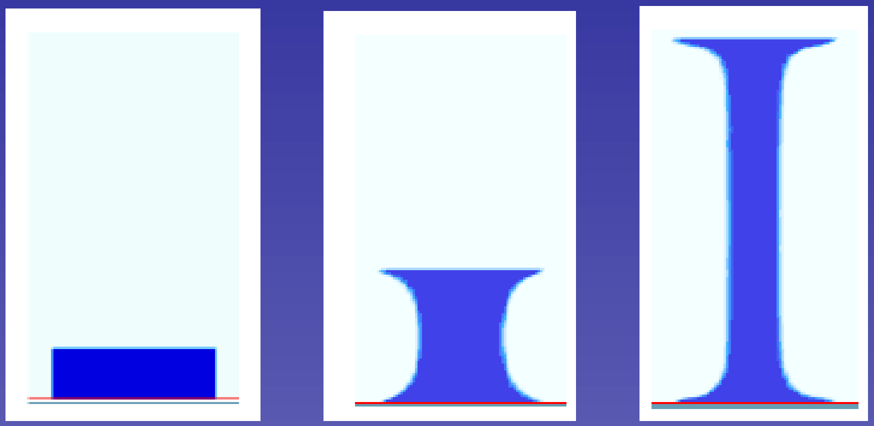
- Local existence and uniqueness (Faedo-Galerkin method + fixed point theorems)
- Convergence of the splitting algorithm : $\mathcal{O}(\Delta t + h)$

Current Work and Perspectives

- Introduction of surface tension

$$-p\mathbf{n} + 2\mu D(\mathbf{u})\mathbf{n} = -(P - P_{\text{atmo}})\mathbf{n} \quad + \quad \sigma\kappa\mathbf{n}.$$

- Coupling with viscoelastic flows (filament stretching).



- Theoretical results in 2D.
- Fast Solvers for Stokes problem.