From Least Squares to Monge-Ampère

Alexandre Caboussat and Roland Glowinski

Department of Mathematics, University of Houston
Houston, TX 77204-3008, USA

caboussat@math.uh.edu
http://www.math.uh.edu/∼caboussat

Graduate Student Seminar
University of Houston
April 10, 2009

Goals

1. Show that 'textbook' techniques, such as least-squares, still have applications nowadays.
2. Provide an example where 'things can go wrong'.
3. Talk about a very important partial differential equation in the field of *fully nonlinear elliptic equations*.

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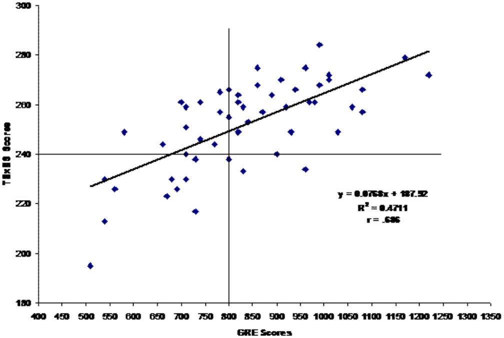
Outline

Starting with Statistics...



For $(x_i, y_i)_{i=1}^N$ given, find the line $y = ax + b$ satisfying

$$\min_{a,b} \sum_{i=1}^{N} (y_i - (ax_i + b))^2 .$$

Linear Regression

Best line to approximate a cloud of point

For $(x_i, y_i)_{i=1}^N$ given, find the line $y = ax + b$ satisfying

$$\min_{a,b} \sum_{i=1}^{N} (y_i - (ax_i + b))^2 .$$

Answer

$$a = \frac{N \sum_{i=1}^N x_i y_i - \left(\sum_{i=1}^N x_i\right)\left(\sum_{i=1}^N y_i\right)}{N\left(\sum_{i=1}^N x_i^2\right) - \left(\sum_{i=1}^N x_i\right)^2},$$

$$b = \frac{\left(\sum_{i=1}^N y_i\right) - a\left(\sum_{i=1}^N x_i\right)}{N}.$$

Approximation of Functions

Problem

For a given function $f \in C^0[a, b]$, find the best polynomial of degree one $P_1(x)$ that approximates the function f on the interval $[a, b]$

Answer: Find the polynomial function $P_1(x)$ that minimizes the distance between f and P_1

$$\min_{P_1 \in \mathbb{P}_1} d(P_1, f).$$

Question: Choice of the distance function?

Distance Function

- L^∞-norm:

$$\min_{a_0,a_1} \max_{x\in[a,b]} \left|f(x) - (a_0 + a_1 x)\right| .$$

This is called a min-max problem.

- L^1-norm:

$$\min_{a_0,a_1} \int_a^b \left|f(x) - (a_0 + a_1 x)\right| dx.$$

Problem: Non-differentiable function.

- L^2-norm:

$$\min_{a_0,a_1} \int_a^b \left(f(x) - (a_0 + a_1 x)\right)^2 dx.$$

Least-squares problem.

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Least-squares problem.

Approximation of Functions by Polynomials

Problem

For a given function $f \in C^0[a,b]$, find the best polynomial of degree n, $P_n(x)$, that approximates the function f on the interval $[a,b]$. Since $P_n(x) = \sum_{k=0}^{n} a_k x^k$, the least squares approach is:

$$\min_{a_0,\ldots,a_n} \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 dx = \min_{P_n \in \mathbb{P}_n} ||f - P_n||^2_{L^2(a,b)}.$$

Normal Equations

The normal equations form the system

$$\frac{\partial}{\partial a_j} ||f - P_n||^2_{L^2(a,b)} = 0, \quad j = 0, \ldots, n$$

In that case:

$$\sum_{i=0}^{n} a_i \int_a^b x^{i+j} dx = \int_a^b f\, x^j dx, \quad j = 0, \ldots, n$$

Approximation of Functions by a Subspace

Let $\{\phi_1, \dots, \phi_n\}$ be a linearly independent family of functions in $C^0[a,b]$. Let

$$V = \text{span}\{\phi_1, \dots, \phi_n\} \subset C^0[a,b], \quad \dim V = n.$$

Problem

For a given function $f \in C^0[a,b]$, find the best function $g \in V$, that approximates the function f on the interval $[a,b]$. This function minimizes the distance between f and g:

$$\min_{g \in V} d(P_1, f).$$

Least-Squares approximations

- For instance, the least squares problem reads:

$$\min_{g\in V}\int_a^b (f(x)-g(x))^2\,dx = \min_{g\in V}\int_a^b \left(f(x)-\sum_{k=1}^{n} a_k\phi_k(x)\right)^2 dx.$$

- The normal equations read:

$$\sum_{k=1}^{n} a_k \int_a^b \phi_k(x)\phi_j(x)dx = \int_a^b f(x)\phi_j(x)dx, \quad j=1,\ldots,n.$$

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Questions

- Choice of the subspace V? What are the basis functions ϕ_k?
- Choice of the distance d? What objective function to minimize?

Dirichlet Problem for the Monge-Ampère Equation

Let $\Omega \subset \mathbb{R}^2$ bounded, smooth, convex. Consider $f \in L^1(\Omega)$, $f > 0$, and $g \in H^{3/2}(\partial\Omega)$.

Find $\psi \in H^2(\Omega)$ satisfying

$$\begin{cases} \det \mathbf{D}^2\psi = f & \text{in } \Omega, \\ \psi = g & \text{on } \Gamma = \partial\Omega. \end{cases}$$

$$\text{where } \mathbf{D}^2\psi = \begin{pmatrix} \dfrac{\partial^2\psi}{\partial x^2} & \dfrac{\partial^2\psi}{\partial x \partial y} \\ \dfrac{\partial^2\psi}{\partial x \partial y} & \dfrac{\partial^2\psi}{\partial y^2} \end{pmatrix}.$$

$$f > 0 \;\Rightarrow\; \psi \text{ convex.}$$

Applications and Interpretation

- "Monge-Ampère equation is in implicit nonlinear elliptic PDEs, the equivalent to the Laplace equation in linear elliptic PDEs".
- Many other applications: Geometry, optimal design of antenna arrays, front formation in meteorology, semigeostrophic flow (slow flows under rotation and stratification), reflector design (scattering field), compressible gas dynamics, etc.
- Optimal transportation problem (Monge problem).

[Caffarelli, Cabre AMS, 1995], [Gutierrez, 2001], [Caffarelli, Milman, 1997].

Optimal Transportation Problem

Discrete case

Given two sets of k points in $\mathbb{R}^n$: $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$, find a mapping of X_k onto Y_j, *i.e.*, among all one-to-one functions $Y(X_k)$ that minimizes some transportation costs, for instance,

$$C = \frac{1}{2} \sum_k |Y(X_k) - X_k|^2 .$$

Continuous case

Find an (admissible) map $Y(X)$, linking two probability densities, that minimizes some cost.
In that case, $Y(X) = \nabla \varphi$, where φ is a convex potential that satisfies the Monge-Ampère equation.

Existence and Regularity of Solutions

- When $f \in L^1(\Omega)$, $f > 0$, and $g \in H^{3/2}(\partial\Omega)$, it makes sense to look for a convex solution $\psi \in H^2(\Omega)$.
- The solution is not necessarily unique (actually at most two solutions).
- The smoothness of the data does no imply the smoothness of the solution. In particular, when the domain Ω is not strictly convex, there is no classical solution (even for f, g and $\partial\Omega$ smooth).

  For instance, if $\Omega = (0,1)^2$, $f = 1$, and $g = 0$, there is no classical solutions $\psi \in C^2(\Omega)$.

  Indeed, $\psi = 0$ on the boundary, implies that both the product $\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2}$ and the cross derivative $\frac{\partial^2 \psi}{\partial x \partial y}$ vanish at the boundary. Which implies that $\det \mathbf{D}^2 \psi$ is strictly smaller than one in a neighborhood of Γ.

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Viscosity Solutions

For conservation laws:

Find u^ε satisfying

$$\begin{cases} -\varepsilon \dfrac{\partial^2 u^\varepsilon}{\partial x^2} + \dfrac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \dfrac{\partial u^\varepsilon}{\partial x} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u^\varepsilon = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

For Hamilton-Jacobi equations:

Find u^ε satisfying

$$\begin{cases} -\varepsilon \Delta u^\varepsilon + \dfrac{\partial u^\varepsilon}{\partial t} + H(\nabla u^\varepsilon, t) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

[L. C. Evans, AMS, 2002]

Viscosity Solutions

- **Moment methods** [Feng, Neilan, 2007]:

Find ψ^ε satisfying

$$\begin{cases} -\varepsilon\Delta^2\psi^\varepsilon + \det \mathbf{D}^2\psi^\varepsilon = f & \text{in } \Omega, \\ \psi^\varepsilon = g & \text{on } \Gamma = \partial\Omega, \\ \Delta^2\psi^\varepsilon = \varepsilon^2, & \text{on } \Gamma = \partial\Omega \end{cases}$$

- We have [Feng, 2009]

$$\lim_{\varepsilon \to 0} \psi^\varepsilon = \psi.$$

- **Finite element approximation.** There are positive constants C, α and β such that

$$||\psi^\varepsilon - \psi|| \leq Ch^\alpha \frac{1}{\varepsilon^\beta}.$$

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Least-Squares Methods

- Introduction of a distance function:

$$\inf_{\varphi \in V_g} J(\varphi),$$

where

$$V_g = \left\{\varphi \in H^2(\Omega),\ \varphi = g\right\},$$

and

$$J(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \left|\ \mathrm{det}\ \mathbf{D}^2\varphi - f\right|^2 dx, & \text{if } \ \mathrm{det}\ \mathbf{D}^2\varphi - f \in L^2(\Omega), \\ +\infty, & \text{otherwise} \end{cases}$$

- Numerical solution with operator-splitting methods and finite elements [Dean, Glowinski, CMAME, 2006].

Least-Squares Methods (2)

- Introduction of a variational principle:

$$\inf_{\varphi \in E_{fg}} \frac{1}{2} \int_\Omega |\Delta \varphi|^2 \, dx,$$

where

$$E_{fg} = \left\{ \varphi \in H^2(\Omega), \ \varphi = g \quad \text{on } \Gamma, \quad \det \mathbf{D}^2 \varphi = f \right\}.$$

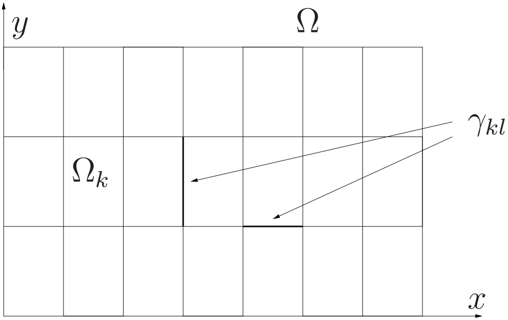
- Numerical solution with penalization, augmented Lagrangian methods and finite elements [Dean, Glowinski, CMAME, 2006].

As we have seen, if we want to use *least-squares* methods, we have many choices for

- The subspace V of approximations;
- The distance (objective function) to minimize.

Let us describe another least-squares approach, a little bit less classical...

Domain Decomposition



The domain Ω is divided into K non-overlapping sub-domains Ω_k, $k = 1, \ldots, K$, such that

$$\overline{\bigcup_{k=1}^{K}} \Omega_k = \overline{\Omega}, \quad \Omega_k \cap \Omega_j = \emptyset \text{ if } j \neq k.$$

Choice of the Subspace

- When looking for *convex solutions*, a convex polynomial function of degree two is associated to each sub-domain Ω_k, namely

$$\psi_k(x,y) = \frac{\sqrt{f_k}}{2}\left(x^2+y^2\right) + \alpha_k x + \beta_k y + \gamma_k,$$

where α_k, β_k and γ_k are unknown real coefficients, and $f_k \simeq f|_{\Omega_k}$, *e.g.*

$$f_k = \frac{\displaystyle\int_{\Omega_k} f(x,y)dxdy}{\displaystyle\int_{\Omega_k} dxdy},$$

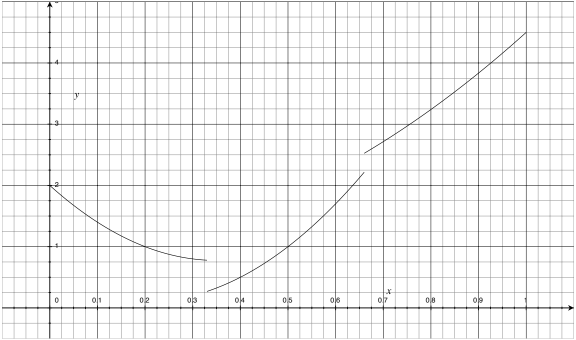
- Note that $\det \mathbf{D}^2\psi_k = f_k \quad \text{in } \Omega_k.$

Choice of the Subspace

$$V = \Big\{ \varphi \in L^2(\Omega),\ \varphi|_{\Omega_k} \in \mathbb{P}_2, \quad \varphi(x,y)|_{\Omega_k} = \frac{\sqrt{f_k}}{2}\left(x^2 + y^2\right) + \alpha_k x + \beta_k y + \gamma_k, \quad \overline{\bigcup_{k=1}^{K} \Omega_k} = \overline{\Omega}, \Big\}$$

Choice of the Objective Function

One dimensional case:



Remember

We are looking for $\psi \in H^2(\Omega)$. In two dimensions, this corresponds to *nearly* $C^1(\Omega)$-functions.

Objective Function

$$\min_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^K} J(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),$$

where

$$J(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \frac{1}{2}\sum_{1\le k,l\le K}\int_{\gamma_{kl}} |\psi_k(x,y)-\psi_l(x,y)|^2\, dS$$
$$+\frac{1}{2}\sum_{1\le k\le K}\int_{\partial\Omega_k\cap\Gamma} |\psi_k(x,y)-g(x,y)|^2\, dS$$
$$+\frac{r}{2}\sum_{1\le k,l\le K}\int_{\gamma_{kl}} \left|\frac{\partial\psi_k}{\partial n_k}(x,y)+\frac{\partial\psi_l}{\partial n_l}(x,y)\right|^2 dS,$$

where $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^K$, $\boldsymbol{\beta} = (\beta_k)_{k=1}^K$ and $\boldsymbol{\gamma} = (\gamma_k)_{k=1}^K$, and $r > 0$.

Alternative choice

$$
\begin{aligned}
J(\alpha,\beta,\gamma) &= \frac{1}{2}\sum_{1\le k,l\le K}\int_{\gamma_{kl}} |\psi_k(x,y)-\psi_l(x,y)|\,dS \\
&+\frac{1}{2}\sum_{1\le k\le K}\int_{\partial\Omega_k\cap\Gamma} |\psi_k(x,y)-g(x,y)|\,dS \\
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\end{aligned}
$$

The Case of a Rectangle

Consider $\Omega = (0, a) \times (0, b)$, I, J given, and $h_x = a/I$, $h_y = b/J$.
For $i = 1, \dots, I$ and $j = 1, \dots, J$, we consider the sub-domain $\Omega_{ij} = ((i-1)h_x, ih_x) \times ((j-1)h_y, jh_y)$ (such that $\overline{\Omega} = \overline{\bigcup_{ij} \Omega_{ij}}$), and the polynomial ψ_{ij} defined by

$$\psi_{ij}(x, y) = \frac{\sqrt{f_{ij}}}{2}\left(x^2 + y^2\right) + \alpha_{ij} x + \beta_{ij} y + \gamma_{ij}.$$

A Collocation Method

- Approximate each integral numerically in

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\end{aligned}
$$

- Numerical integration with Gauss-Legendre points

$$\int_a^b g(x)dx \simeq \sum_{k=1}^{n}\omega_k g(x_k).$$

Numerical integration is exact (if n large enough, *i.e.* $n=3$)!

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System of Equations

Matching of Boundary conditions:

$$\alpha_{i1} x_{ki} + \gamma_{i1} = g_{x,ki1} - \frac{\sqrt{f_{i1}}}{2} x_{ki}^2, \qquad k = 1, n, \quad i = 1, I.$$

$$\alpha_{iJ} x_{ki} + \beta_{iJ} b + \gamma_{iJ} = g_{x,ki2} - \frac{\sqrt{f_{iJ}}}{2} \left(x_{ki}^2 + b^2 \right), \quad k = 1, n, \quad i = 1, I.$$

$$\beta_{1j} y_{kj} + \gamma_{1j} = g_{y,k1j} - \frac{\sqrt{f_{1j}}}{2} y_{kj}^2, \qquad k = 1, n, \quad j = 1, J.$$

$$\alpha_{Ij} a + \beta_{Ij} y_{kj} + \gamma_{Ij} = g_{y,k2j} - \frac{\sqrt{f_{Ij}}}{2} \left(a^2 + y_{kj}^2 \right), \quad k = 1, n, \quad j = 1, J.$$

System of Equations (2)

Matching of the solution on interior interfaces:

$$(\alpha_{ij+1} - \alpha_{ij})x_{ki} + (\beta_{ij+1} - \beta_{ij})(jh_y) + (\gamma_{ij+1} - \gamma_{ij}) =$$
$$\frac{1}{2}\left(\sqrt{f_{ij}} - \sqrt{f_{ij+1}}\right)\left(x_{ki}^2 + (jh_y)^2\right), \quad k = 1, n,\ j = 1, J-1,\ i = 1, I.$$

$$(\alpha_{i+1j} - \alpha_{ij})(ih_x) + (\beta_{i+1j} - \beta_{ij})y_{kj} + (\gamma_{i+1j} - \gamma_{ij}) =$$
$$\frac{1}{2}\left(\sqrt{f_{ij}} - \sqrt{f_{i+1j}}\right)\left((ih_x)^2 + y_{kj}^2\right), \quad k = 1, n,\ j = 1, J,\ i = 1, I-1.$$

Matching of the normal derivatives on interior interfaces:

$$r\beta_{ij+1} - r\beta_{ij} = r\left(\sqrt{f_{ij}} - \sqrt{f_{ij+1}}\right)(jh_y), \quad j = 1, J-1,\ i = 1, I.$$
$$r\alpha_{i+1j} - r\alpha_{ij} = r\left(\sqrt{f_{ij}} - \sqrt{f_{i+1j}}\right)(ih_x), \quad j = 1, \dots, J,\ i = 1, I-1.$$

Conclusion

$3IJ$ unknowns

$2n(I+J)+n[I(J-1)+J(I-1)]+I(J-1)+J(I-1)$ equations

Over-determined system!

$$\mathbf{A}\mathbf{x}=\mathbf{b}.$$

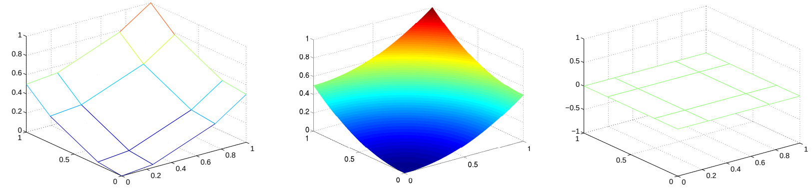
The matrix $\mathbf{A}$ is of full column rank. The solution of an over-determined system is addressed by the normal equations:

$$\mathbf{A}^T\mathbf{A}\mathbf{x}=\mathbf{A}^T\mathbf{b}.$$

Validation Example

$$\psi(x, y) = \frac{1}{2}\left(x^2 + y^2\right), \; (x, y) \in \Omega.$$

i.e. $f(x, y) = 1$, and $g(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$.

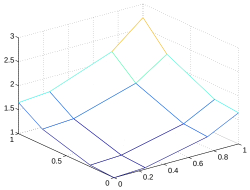


Approximation is exact! $(\alpha = \beta = \gamma = 0)$

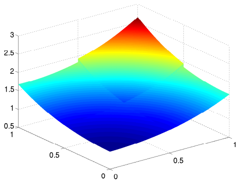
Exponential Example

$$\psi(x,y) = e^{\frac{1}{2}(x^2+y^2)}, \ (x,y) \in \Omega.$$

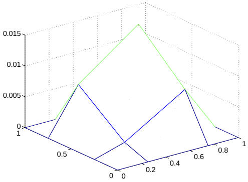
i.e. $f(x,y) = \left(1 + (x^2+y^2)\right) e^{(x^2+y^2)}, \; g(x,y) = e^{\frac{1}{2}(x^2+y^2)}.$



Exact



Approximation

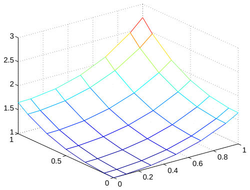


Error

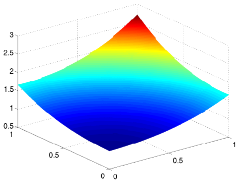
Exponential Example

$$\psi(x,y) = e^{\frac{1}{2}(x^2+y^2)},\ (x,y) \in \Omega.$$

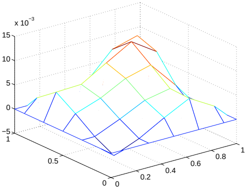
$$i.e.\ \ f(x,y) = \left(1 + (x^2+y^2)\right) e^{(x^2+y^2)},\ g(x,y) = e^{\frac{1}{2}(x^2+y^2)}.$$



Exact



Approximation



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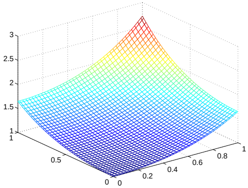
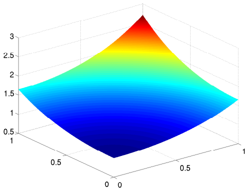
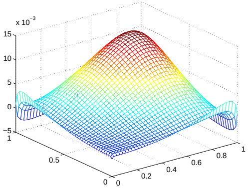


Exact



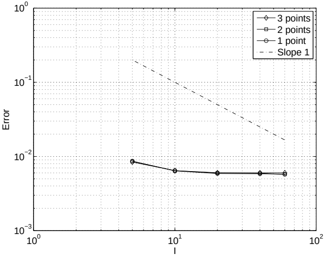
Approximation



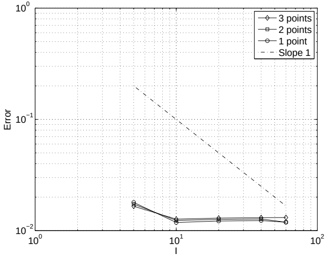
Error

Convergence of Approximations

$I = J$	$\|\|\psi_h - \psi\|\|_{L^2(\Omega)}$	$\|\|\psi_h - \psi\|\|_{L^\infty(\Omega)}$
5	8.405588e-03	1.656257e-02
10	6.439176e-03	1.268631e-02
20	6.045698e-03	1.296824e-02
50	6.030157e-03	1.311007e-02



L^2-norm



L^∞-norm

A Second Example

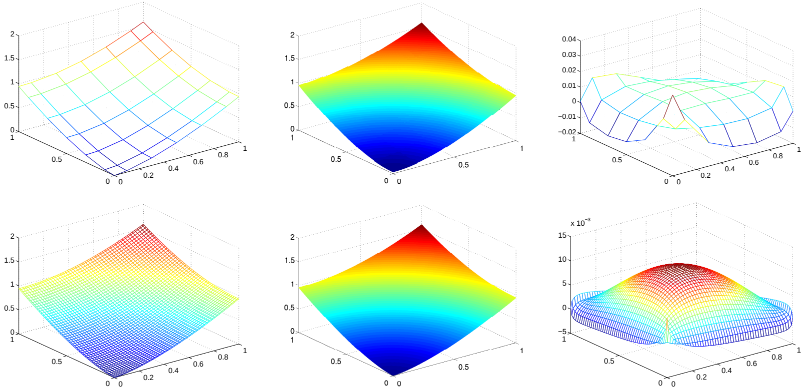
$$\psi(x,y) = \frac{\left(2\sqrt{x^2+y^2}\right)^{3/2}}{3}, \ (x,y) \in \Omega.$$

i.e. $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$, $g(x,y) = \frac{\left(2\sqrt{x^2+y^2}\right)^{3/2}}{3}$.

$I = J$	$\|\|\psi_h - \psi\|\|_{L^2(\Omega)}$	$\|\|\psi_h - \psi\|\|_{L^\infty(\Omega)}$
5	9.319363e-03	3.242689e-02
10	6.628609e-03	2.319420e-02
20	6.116831e-03	1.483774e-02
50	6.113496e-03	1.115342e-02

A Second Example (Results)

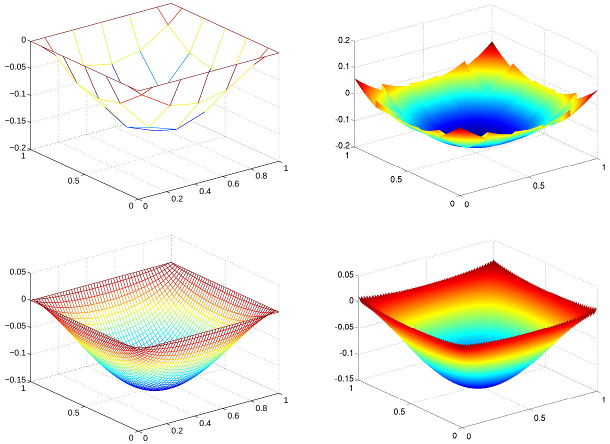
$l = 5 / l = 50$



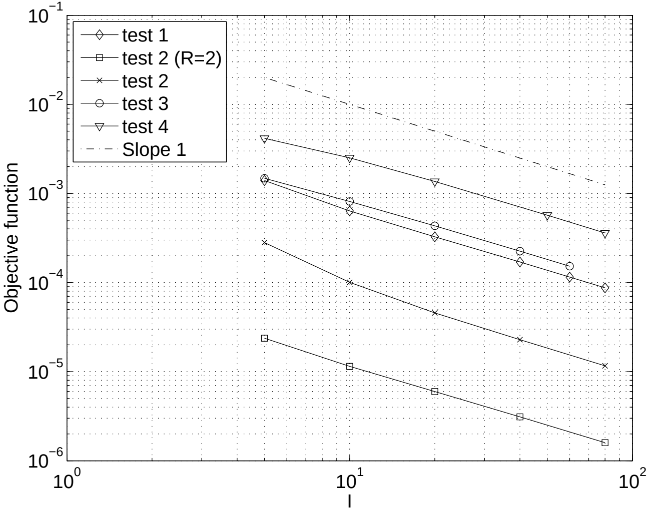
Exact Approximation Error

Example with No Smooth Solution

$$f(x,y) = 1, \quad g(x,y) = 0.$$



Decrease of Objective Function



Conclusions

- The distance decreases to zero when the number of sub-domains increases. In some sense, the approximation gets better.
- The error between approximation and exact functions stagnates. In some sense, the approximation does not get better.

- The sequence of subspaces V does not converge to $H^2(\Omega)$ when we increase the number of sub-domains. This is a lack of density.
- However the method does converge to some function, which is the best approximation of ψ in V (some projection of ψ on V).

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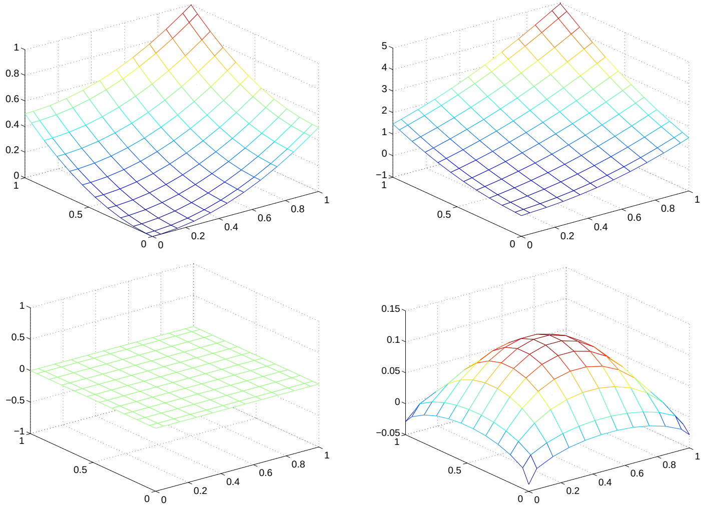
Influence of the Eigenvalues of the Hessian



$\psi(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$ $\qquad$ $\psi(x,y) = \frac{1}{2}\left(x^2 + y^2 + 2xy\right)$

Remedy?

Expand the approximation subspace!

$$\psi_k(x,y) = \frac{\sqrt{f_k}}{2}\left(a_k x^2 + 2b_k xy + c_k y^2\right) + \alpha_k x + \beta_k y + \gamma_k,$$

with α_k, β_k and γ_k as before, and a_k, b_k and c_k verifying

$$a_k b_k - c_k^2 = 1.$$

Justification: Local second order Taylor expansion ψ.

Other Examples

- Find $\mathbf{u} \in (H^1(\Omega) \cap L^\infty(\Omega))^2$ satisfying

$$\nabla \cdot \mathbf{u} = f (\in L^1(\Omega))$$

$$\min_{\mathbf{u} \in \mathbf{S}_f} \frac{1}{2} \int_\Omega |\nabla \mathbf{v}|^2 \, dx + g \, ||\mathbf{v}||_\infty$$

$$\mathbf{S}_f = \left\{ \mathbf{v} \in (H^1(\Omega) \cap L^\infty(\Omega))^2, \; \nabla \cdot \mathbf{u} = f \right\}.$$

Other Examples

- Find the best constants in Sobolev injections:

$$||\varphi||_{L^\infty(\Omega)} \leq C\,||\varphi||_{H^2(\Omega)}\,,\ \forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega),\ \Omega \text{ convex}\ .$$

$$\sup_{\varphi \in \Sigma} ||\varphi||_{L^\infty(\Omega)}\,,$$

$$\Sigma = \left\{ \psi \in H^2(\Omega) \cap H^1_0(\Omega),\ ||\Delta\psi||_{L^2(\Omega)} = 1 \right\}.$$

Other Examples

- Numerical simulation involving non-smooth operators:

$$\frac{\partial u}{\partial t} + \partial I_K(u) \ni f$$

$$K = \left\{ v \in H_0^1(\Omega),\ |\nabla v| \leq 1,\ v \geq u_0 \right\}.$$

$$I_K(v) = \begin{cases} 0, & \text{if } v \in K, \\ +\infty, & \text{if } v \notin K, \end{cases}$$


MERRIE MELODIES
"That's all Folks!"
A WARNER BROS. CARTOON