TRANSONIC REGULAR REFLECTION FOR THE NONLINEAR WAVE SYSTEM

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Abstract. We consider Riemann data for the nonlinear wave system which result in a regular reflection with a subsonic state behind the reflected shock. The problem in self-similar coordinates leads to a system of mixed type and a free boundary value problem for the reflected shock and the solution in the subsonic region. We show existence of a solution in a neighborhood of the reflection point.

1. Introduction

In this paper we continue the program initiated by Čanić, Keyfitz, Kim and Lieberman on solving Riemann problems for two-dimensional systems of hyperbolic conservation laws modeling shock reflection. The first step in our approach is to write the system in self-similar coordinates and obtain a system which changes type. One finds a solution in the hyperbolic part of the domain using the standard theory of one-dimensional hyperbolic conservation laws and the notion of quasi-one-dimensional Riemann problems developed by Čanić, Keyfitz and Kim (see [2] for the unsteady small disturbance equation, [5] for the nonlinear wave system and [3] for a general discussion). The position of the reflected shock is formulated as a free boundary problem coupled to the subsonic state behind the shock through the Rankine-Hugoniot conditions. To solve the free boundary problem behind the reflected shock, one proceeds as follows: (1) fix a curve within a certain bounded set of admissible curves approximating the free boundary, (2) solve the fixed boundary problem, and (3) update the position of the reflected shock. This gives a mapping on the set of admissible curves, and one proves there is a fixed point in a weighted Hölder space.

The idea was first implemented on a shock perturbation problem for the steady transonic small disturbance equation by Čanić, Keyfitz and Lieberman [8]. It was extended to two types of regular reflection for the unsteady transonic small disturbance equation in Čanić, Keyfitz and Kim [4] (transonic regular reflection) and Čanić, Keyfitz and Kim [6] (supersonic regular reflection). The principal features of this method for a class of two-dimensional conservation laws (including the unsteady transonic small disturbance equation, the nonlinear wave system, and the isentropic gas dynamics equations) are presented in the survey paper by Keyfitz [16]. A detailed study of the subsonic solution to the fixed boundary problem for a class of operators satisfying certain structural conditions is given in [15] by Jegdić, Keyfitz and Čanić.

In this paper we consider the two-dimensional nonlinear wave system (NLWS). A partial solution to a Riemann problem for the NLWS leading to Mach reflection is given in [7] by Čanić, Keyfitz and Kim. Solving a regular reflection problem for this system extends the results of Čanić, Keyfitz and Kim [4, 6] to a more complicated equation and boundary condition, and sets the stage for a further task (not attempted in this paper), obtaining a global solution to a Riemann problem. We also present an improved way of handling the artificial far-field Dirichlet boundary condition. We take advantage of the simplified form of the NLWS in polar coordinates. Change of variables to self-similar and polar coordinates is given in [5], as well as the explicit solutions to quasi-one-dimensional Riemann problems that we use here.

1.1. **Related Work.** An overview of oblique shock wave reflection in steady, pseudo-steady and unsteady flows from a phenomenological point of view is given in [1] by Ben-Dor. Existence and stability of steady multidimensional transonic shocks was studied by Chen and Feldman in [10]-[13]. Zheng [26] proved existence of a global solution to a weak regular reflection for the pressure gradient system. Two-dimensional Riemann problems for isentropic and polytropic gas dynamics equations were studied by Zhang and Zheng in [23]. They give conjectures on the structure of the solutions when initial data is posed in four quadrants and each jump results in exactly one planar shock, rarefaction wave or slip plane far from origin. General mathematical theory of two-dimensional Riemann problem for both scalar equations and systems is presented by Zheng in [24]. An approach for proving existence of a global solution to a weak regular reflection for polytropic gas dynamics equations with large gas constant gamma is given in [25] by Zheng. Serre derives maximum principle for the pressure and other a priori estimates in [22]. We also mention an earlier work of Chang and Chen [9] on a formulation of a free boundary problem resulting from a weak regular reflection for the polytropic gas dynamics equations.

1.2. **Summary of the Results.** In Section 2 we state a Riemann problem for the NLWS resulting in a regular reflection with a subsonic state behind the reflected shock. The discussion on how the initial data is chosen so that the configuration leads to this type of reflection is given in Appendix A. We write the problem in self-similar coordinates. Along the lines of the study in [7], we find a solution in the hyperbolic part of the domain, derive the equation of the reflected shock and give a formulation of the free boundary problem behind the reflected shock. Our main result, Theorem 2.3, is local existence of a solution to this free boundary problem, and the rest of the paper is devoted to its proof.

In Section 3 we reformulate the problem using a second order elliptic equation and from the Rankine-Hugoniot conditions along the free boundary we obtain an oblique derivative boundary condition and an equation describing the position of the reflected shock. To ensure that the problem is well-defined we introduce several cut-off functions. This gives the modified free boundary problem stated in Theorem 3.1.

The first step in proving Theorem 3.1 is, as outlined above, to fix the position of the free boundary within a bounded set of admissible curves and to solve the modified fixed boundary problem. This task is completed in Section 4. We use the study in [15] of fixed boundary value problems for a class of operators which satisfy certain structural assumptions. For convenience we list those structural conditions in the notation of this paper in Appendix B.

In Section 5 we use the Schauder fixed point theorem to show existence of a solution to the modified free boundary problem.

Finally, the conditions under which a solution of the modified free boundary problem solves the original free boundary problem are discussed in Section 6, completing the proof of Theorem 2.3.

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2. The Statement of the Free Boundary Problem

In this section we formulate a Riemann problem leading to transonic regular reflection for the two-dimensional NLWS. The problem is considered in self-similar coordinates, yielding a system which changes type. We find a solution in the hyperbolic part of the domain and formulate the problem for the position of the reflected shock. The main result of the paper is local existence of a solution in the subsonic part of the domain and is stated in Theorem 2.3.

The two-dimensional NLWS is a hyperbolic system of three conservation laws:

$$\begin{aligned} \rho_t + m_x + n_y &= 0, \\ m_t + p_x &= 0, \qquad (t,x,y) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}. \\ n_t + p_y &= 0, \end{aligned} \tag{2.1}$$

Here, $\rho : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to (0,\infty)$ stands for the density; $m, n : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are "momenta" in the x and y directions, respectively; and $p = p(\rho)$ is the pressure. We denote $c^2(\rho) := p'(\rho)$, and we require that $c^2(\rho)$ be a positive and increasing function for all $\rho > 0$.

We consider symmetric Riemann initial data (Figure 1) consisting of two sectors separated by the half lines $x = ky$ and $x = -ky$, $y \geq 0$, with $k > 0$. The data are

$$U(0,x,y) = \begin{cases} U_0 = (\rho_0, 0, n_0), & -ky < x < ky,\ y > 0, \\ U_1 = (\rho_1, 0, 0), & \text{otherwise}, \end{cases} \tag{2.2}$$

with the assumption $\rho_0 > \rho_1 > 0$. The constants k and n_0 are specified in Section 2.1 (as described in Appendix A) in terms of ρ_0 and ρ_1 so that the Riemann problem (2.1), (2.2) results in a regular reflection and we choose a solution (when there is more than one) with a subsonic state behind the reflected shock.

Note that we can eliminate m and n in (2.1) and obtain a second order equation for ρ alone:

$$\rho_{tt} = -m_{tx} - n_{ty} = p_{xx} + p_{yy} = \mathrm{div}(p_x, p_y) = \mathrm{div}(c^2(\rho)\nabla\rho), \tag{2.3}$$

where "div" stands for the divergence and ∇ for the gradient in spatial variables.

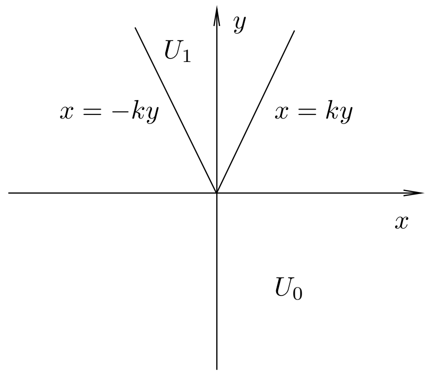

FIGURE 1. The Riemann Initial Data

We introduce self-similar coordinates $\xi = x/t$ and $\eta = y/t$, and obtain

$$\begin{aligned} -\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta &= 0, \\ -\xi m_\xi - \eta m_\eta + p_\xi &= 0, \\ -\xi n_\xi - \eta n_\eta + p_\eta &= 0, \end{aligned} \tag{2.4}$$

from (2.1), and the second order equation

$$((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta)_\xi + ((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0, \tag{2.5}$$

from equation (2.3). It is clear that when the equation (2.5) is linearized about a constant state $\rho > 0$, the equation changes type across the sonic circle

$$C_\rho : \xi^2 + \eta^2 = c^2(\rho).$$

More precisely, (2.5) is hyperbolic outside of the circle C_ρ and is elliptic inside.

2.1. **Solution in the Hyperbolic Part of the Domain.** Suppose that the densities $\rho_0 > \rho_1 > 0$ are given. In this section we specify k and n_0, in terms of ρ_0 and ρ_1, so that a transonic regular reflection occurs, and we find a solution to the Riemann problem (2.1), (2.2) in the hyperbolic region.

The parameter $n_0 = n_0(\rho_0, \rho_1, k)$ is chosen so that each of the two discontinuities $x = \pm ky$, $y \geq 0$, is resolved as a shock and a linear wave far from the origin (Figure 2). From the calculation in [5, Appendix A] this means that given $\rho_0 > \rho_1 > 0$ and $k > 0$, we take

$$n_0 = \frac{\sqrt{1+k^2}}{k}\sqrt{(p(\rho_0) - p(\rho_1))(\rho_0 - \rho_1)}. \tag{2.6}$$

Using the Rankine-Hugoniot relations, the one-dimensional Riemann solution with states U_0 on the left and U_1 on the right consists of a linear wave $l_a : \xi = k\eta$, an intermediate state $U_a = (\rho_0, m_a, n_a)$ and a shock $S_a : \xi = k\eta + \chi_a$, with

$$\chi_a = -\sqrt{1+k^2}\sqrt{\frac{[p]}{[\rho]}}, \quad m_a = -\sqrt{\frac{[p][\rho]}{1+k^2}}, \quad n_a = -km_a, \tag{2.7}$$

where $[\cdot]$ denotes the jump between the states U_0 and U_1. By symmetry, the one-dimensional solution in the left half-plane consists of a shock $S_b : \xi = -k\eta - \chi_a$, an

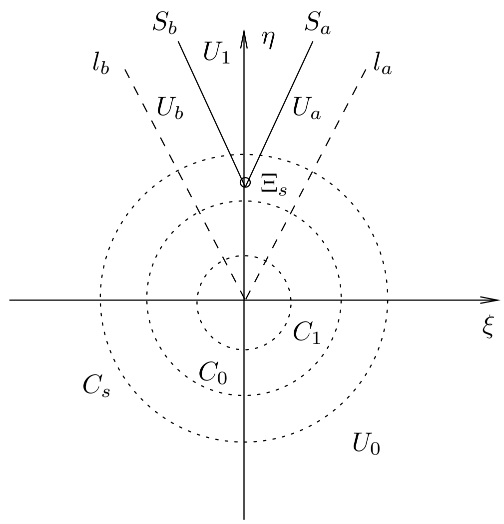

FIGURE 2. Interactions in the hyperbolic region

intermediate state $U_b = (\rho_0, -m_a, n_a)$ and a linear wave $l_b : \xi = -k\eta$. Note that the sonic circles for the states U_a and U_b coincide with the sonic circle for U_0

$$C_0 : \xi^2 + \eta^2 = c^2(\rho_0).$$

The first restriction on the choice of $k = k(\rho_0, \rho_1)$ is that the point $\Xi_s = (0, \eta_s)$ where the shocks S_a and S_b meet should lie above the circle C_0. We find

$$\eta_s = \frac{1}{k}\sqrt{\frac{(1+k^2)(p(\rho_1)-p(\rho_0))}{\rho_1-\rho_0}}. \tag{2.8}$$

Since the point Ξ_s is hyperbolic with respect to U_a and U_b, we solve a quasi-one-dimensional Riemann problem at Ξ_s with states U_a and U_b, on the left and on the right, respectively (with respect to an observer facing the origin) along a line segment through Ξ_s which is parallel to the ξ-axis. A further restriction on the value of $k = k(\rho_0, \rho_1)$ is that this quasi-one-dimensional Riemann problem have a solution (for details see Appendix A). In short, given $\rho_0 > \rho_1 > 0$, there exists a value $k_C(\rho_0, \rho_1)$ with the property that if k is chosen so that

$$0 < k < k_C, \tag{2.9}$$

then the point Ξ_s is above the sonic circle C_0 and, moreover, the quasi-one-dimensional Riemann problem at Ξ_s with states U_a and U_b, on the left and on the right, respectively, has a solution. From now on, we assume that the densities $\rho_0 > \rho_1 > 0$ are fixed, that the parameter k is such that (2.9) holds and that the momentum n_0 is chosen as in (2.6).

Further, a calculation in Appendix A shows that if a solution to the above quasi-one-dimensional Riemann problem at the reflection point Ξ_s exists, there usually are two such solutions. Both consist of a shock connecting the state U_a to an intermediate state and a shock connecting this intermediate state to U_b. Let us denote the intermediate states for these two solutions by

$$U_R = (\rho_R, m_R, n_R) \quad \text{and} \quad U_F = (\rho_F, m_F, n_F).$$

More precisely (see Appendix A), we have

$$\rho_R, \rho_F > \rho_0, \quad m_R = m_F = 0, \tag{2.10}$$

and we choose $\rho_R < \rho_F$. We find that $c(\rho_F) > \eta_s$ for all $k \in (0, k_C)$, and that $c(\rho_R) > \eta_s$ only when k is large enough, say $k \in (k_*, k_C)$, for some value $k_*(\rho_0, \rho_1)$. Therefore, the reflection point Ξ_s is subsonic with respect to the state U_F for all $k \in (0, k_C)$, and Ξ_s is subsonic with respect to the state U_R if $k \in (k_*, k_C)$. We denote the value of our solution at the reflection point Ξ_s by $U_s = (\rho_s, m_s, n_s)$, and we choose

$$U_s := U(\Xi_s) = \begin{cases} U_R \quad \text{or} \quad U_F, & k \in (k_*, k_C), \\ U_F, & k \in (0, k_*]. \end{cases} \tag{2.11}$$

This implies that the point Ξ_s is inside the sonic circle

$$C_s : \xi^2 + \eta^2 = c^2(\rho_s).$$

As a consequence, the reflected shocks we study here are transonic throughout their length. By causality, they cannot exit the sonic circle C_s and, by the Lax admissibility condition (see [2] for the equivalent discussion on the unsteady transonic small disturbance equation), they also do not cross the sonic circle C_0 (Figure 3).

Remark 2.1. If the two reflected shocks at the point Ξ_s were rectilinear, by the Rankine-Hugoniot relations, their equations would be

$$\eta = \eta_s \pm \xi \sqrt{\frac{\eta_s^2}{(p(\rho_s) - p(\rho_0))/(\rho_s - \rho_0)} - 1}.$$

2.2. **Position of the Reflected Shock.** Since the Riemann problem presented above is symmetric with respect to the η-axis, from now till the end of the paper we restrict our attention to the right half plane $\{(\xi, \eta) : \xi \geq 0\}$. Writing (2.4) in polar coordinates

$$r = \sqrt{\xi^2 + \eta^2} \quad \text{and} \quad \theta = \arctan(\eta/\xi),$$

we obtain

$$\begin{pmatrix} -r & \cos\theta & \sin\theta \\ c^2(\rho)\cos\theta & -r & 0 \\ c^2(\rho)\sin\theta & 0 & -r \end{pmatrix} U_r + \frac{1}{r} \begin{pmatrix} 0 & -\sin\theta & \cos\theta \\ -c^2(\rho)\sin\theta & 0 & 0 \\ c^2(\rho)\cos\theta & 0 & 0 \end{pmatrix} U_\theta = 0,$$

or, in conservation form,

$$\partial_r \begin{pmatrix} -r\rho + m\cos\theta + n\sin\theta \\ p(\rho)\cos\theta - rm \\ p(\rho)\sin\theta - rn \end{pmatrix} + \partial_\theta \begin{pmatrix} \frac{1}{r}(-m\sin\theta + n\cos\theta) \\ -\frac{p(\rho)}{r}\sin\theta \\ \frac{p(\rho)}{r}\cos\theta \end{pmatrix} = \begin{pmatrix} -\rho - \frac{1}{r}(m\cos\theta + n\sin\theta) \\ -m - \frac{p(\rho)}{r}\cos\theta \\ -n - \frac{p(\rho)}{r}\sin\theta \end{pmatrix}.$$

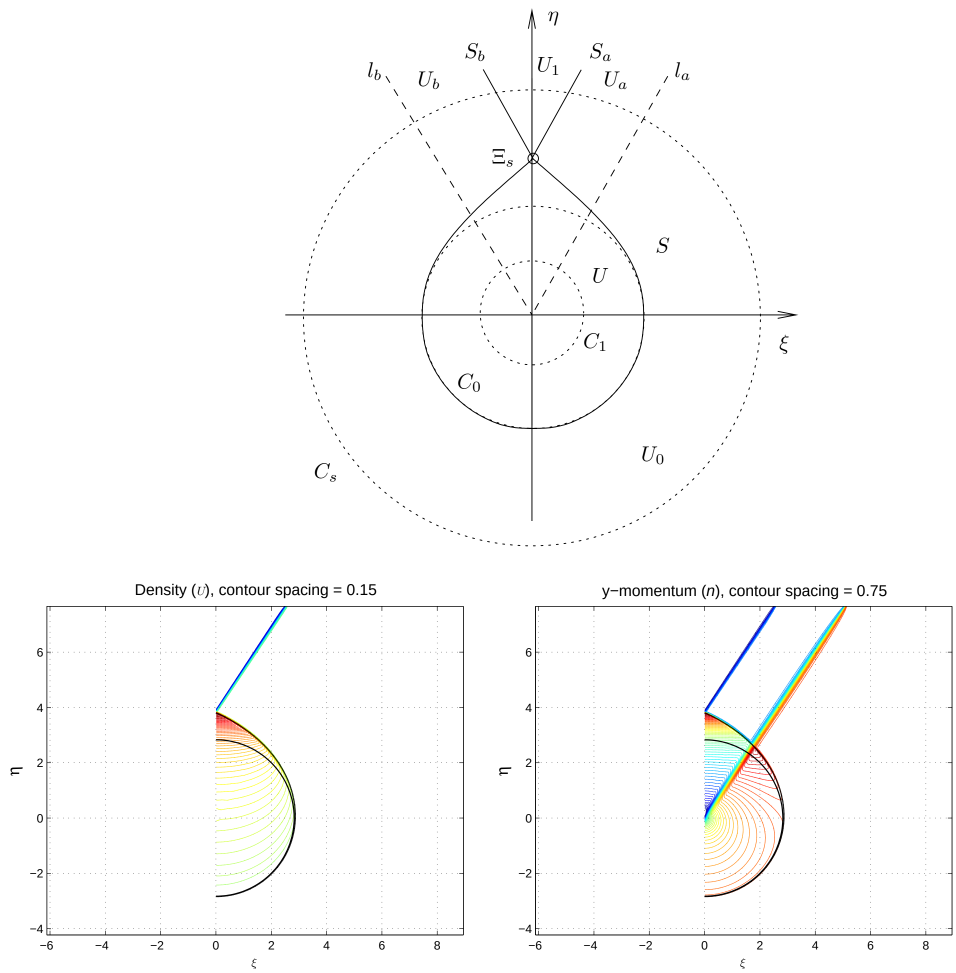

FIGURE 3. Transonic regular reflection for the NLWS: definition of the states (top) and numerical simulation showing the contour plot of ρ (bottom left) and the contour plot of n (bottom right). The inner circle on the bottom figures corresponds to the sonic circle C_0, and the curve following the reflected wave corresponds to the numerically calculated transition between supersonic and subsonic flow.

Let $S : r = r(\theta), \theta \in [-\pi/2, \pi/2]$, denote the reflected transonic shock in the right-half plane. The Rankine-Hugoniot relations along S are

$$\begin{aligned} -r[\rho] + [m]\cos\theta + [n]\sin\theta &= \frac{dr}{d\theta}\frac{1}{r}\left(-[m]\sin\theta + [n]\cos\theta\right) \\ [p]\cos\theta - r[m] &= [p] - \frac{dr}{d\theta}\frac{\sin\theta}{r} \\ [p]\sin\theta - r[n] &= [p]\frac{dr}{d\theta}\frac{\cos\theta}{r}, \end{aligned} \tag{2.12}$$

where $U = (\rho, m, n)$ stands for the unknown solution behind the reflected shock and $[\,\cdot\,]$ now denotes the jump between the states U_0 and U. We express $[m]$ and $[n]$ from the second and the third equations in (2.12), respectively, and substitute into the first equation to obtain

$$\frac{dr}{d\theta} = r\sqrt{\frac{r^2}{s^2} - 1}, \tag{2.13}$$

with

$$s^2 := \frac{p(\rho_0) - p(\rho)}{\rho_0 - \rho}. \tag{2.14}$$

Notice that this shock evolution equation is independent of m and n.

We recall the properties of s, from [7]

Lemma 2.2. *Define the function*

$$s(a,b) := \begin{cases} \sqrt{\frac{p(a)-p(b)}{a-b}}, & a, b > 0,\ b \neq a \\ c(a), & b = a \end{cases} \tag{2.15}$$

Then

(a) for fixed $b > 0$, the $s(\cdot, b)$ is increasing on $(0, \infty)$,
(b) $\lim_{b \to a} s(a,b) = c(a)$, for $a > 0$, and
(c) if $a > b > 0$, then $s(a,b) < c(a)$.

2.3. **The Statement of the Main Result.** In this section we formulate the free boundary problem behind the reflected shock.

For the reasons explained in Section 3.2, we must exclude from our analysis the point $r(-\frac{\pi}{2})$ where the reflected shock intersects the $\eta-$axis (Ξ_0 in Figure 4). For this reason, throughout the paper we fix an angle $\theta^* \in (-\pi/2, \pi/2)$. We denote the intersection of the reflected shock S and the line $\{(r, \theta^*) : r > 0\}$ by V, and define the closed line segment $\sigma = [O, V]$, where O is the origin; the vertical open line segment $\Sigma_0 = (O, \Xi_s)$; and the open curve

$$\Sigma = \{(r(\theta), \theta) : \theta \in (\theta^*, \pi/2)\}.$$

The domain whose boundary is $\Xi_s \cup \Sigma \cup \sigma \cup \Sigma_0$ is denoted by Ω.

We will impose a Dirichlet boundary condition for ρ along σ.

First, we define the set $\mathcal{K}$ of admissible shock curves. Suppose that $\rho_0 > \rho_1 > 0$ and $k \in (0, k_C(\rho_0, \rho_1))$ are fixed. Let the parameter $\theta^* \in (-\pi/2, \pi/2)$ be arbitrary. We define the set $\mathcal{K}$ of candidate functions $r(\theta)$, $\theta \in [\theta^*, \pi/2]$, describing the free boundary Σ, by the following four properties.

- *smoothness:*
$$r(\theta) \in H_{1+\alpha_{\mathcal{K}}},$$
where $\alpha_{\mathcal{K}} \in (0,1)$ will be chosen later and $H_{1+\alpha_{\mathcal{K}}}$ is the Holder space defined in Appendix C,
- *conditions at the end point Ξ_s:*
$$r(\pi/2) = \eta_s \quad \text{and} \quad r'(\pi/2) = \eta_s\sqrt{\frac{\eta_s^2}{s^2(\rho_s, \rho_0)} - 1},$$
(the second condition comes from Remark 2.1)
- *boundedness:*
$$L \le r(\theta) \le \eta_s, \quad \theta \in (\theta^*, \pi/2),$$

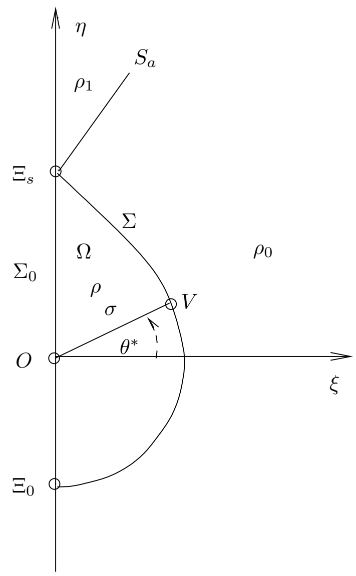

FIGURE 4. The domain Ω and its boundary.

- *monotonicity:*

$$L\sqrt{\delta_*} \le r'(\theta) \le \eta_s \sqrt{\frac{\eta_s^2}{c^2(\rho_0)} - 1}, \quad \theta \in (\theta^*, \pi/2), \tag{2.16}$$

where $\delta_* > 0$ will be specified later in terms of the fixed parameters ρ_0, ρ_1 and k.

A value of L we can use in this paper is

$$L := \frac{\eta_s}{e^{(\pi/2-\theta^*)\sqrt{\eta_s^2/c^2(\rho_0)-1}}}.$$

We show this is an appropriate value in the proof of Lemma 5.1.

Although it is convenient to define the free boundary Σ by a curve $r = r(\theta)$ in polar coordinates, we sometimes write Σ as $\xi = \xi(\eta)$ in self-similar Cartesian coordinates.

On σ we impose an artificial Dirichlet condition, $\rho(r,\theta^*) = f(r)$, chosen so that ρ is larger than its value ρ_0 outside Σ, and so that U is subsonic along σ. (These are the properties that the global solution is expected to have along such a curve.) Let $\epsilon_* \in (0, \rho_s - \rho_0)$ be fixed throughout the paper and let $f : [0, \eta_s] \to \mathbb{R}$ be a function in the Holder space $H_{\gamma;(0,\eta_s)}$, for a parameter $\gamma \in (0,1)$ to be determined later, such that

$$\rho_0 + \epsilon_* \le f(r) \le \rho_s, \qquad c^2(f(r)) > r^2, \qquad 0 \le r \le \eta_s. \tag{2.17}$$

With this notation we can now state the main result.

Theorem 2.3. *(Free boundary problem)*

Let the parameters $\rho_0 > \rho_1 > 0$ and $k \in (0, k_C(\rho_0,\rho_1))$ be fixed. For every $\theta^* \in (-\pi/2, \pi/2)$ and $\epsilon_* \in (0, \rho_s - \rho_0)$, there exists $\gamma_0 > 0$, depending on ρ_0, ρ_1, θ^*

and ϵ_*, such that for any $\gamma \in (0, \min\{1, \gamma_0\})$, $\alpha_{\mathcal{K}} = \gamma/2$ and any function $f \in H_\gamma$ satisfying (2.17), the free boundary problem for ρ, m, n and r given by

$$\left.\begin{array}{rcl} -\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta &=& 0 \\ -\xi m_\xi - \eta m_\eta + p_\xi &=& 0 \\ -\xi n_\xi - \eta n_\eta + p_\eta &=& 0 \end{array}\right\} \qquad in \quad \Omega,$$

$$\left.\begin{array}{rcl} -r[\rho] + [m]\cos\theta + [n]\sin\theta &=& \frac{dr}{d\theta}\frac{1}{r}\left(-[m]\sin\theta + [n]\cos\theta\right) \\ [p]\cos\theta - r[m] &=& -[p]\frac{dr}{d\theta}\frac{\sin\theta}{r} \\ [p]\sin\theta - r[n] &=& [p]\frac{dr}{d\theta}\frac{\cos\theta}{r} \end{array}\right\} \qquad on \quad \Sigma,$$

$$r(\pi/2) = \eta_s,$$

$$\rho = f \quad on \quad \sigma, \qquad \rho_\xi = 0 \quad on \quad \Sigma_0, \qquad \rho(\Xi_s) = \rho_s,$$

has a solution $\rho, m, n \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ and $r \in H_{1+\alpha_{\mathcal{K}}}$ in a finite neighborhood of the reflection point Ξ_s.

3. Derivation of the Modified Free Boundary Problem

Our main tool in proving Theorem 2.3 is the Hölder theory of second order elliptic equations, developed and expounded by Gilbarg, Trudinger and Lieberman. As noted, we can reformulate the first order system in ρ, m and n (the subject of Theorem 2.3) as a second order equation in ρ, (2.5), and in Section 3.1 we introduce a cut-off function to keep the second-order equation strictly elliptic. Further, instead of posing the Rankine-Hugoniot conditions (2.12) along the reflected shock, we derive an oblique derivative boundary condition for ρ on Σ in Section 3.2. We introduce a further cut-off function to ensure that the derivative boundary operator on Σ is oblique. In Section 3.3 we modify the shock evolution equation (2.13) for the reflected shock S, to ensure that it is well-defined. Thus, we obtain a problem that does not involve m or n. (Towards the end of the paper, in Section 5, we show how to recover m and n from the second and third equations in (2.4) by integrating along the radial direction.) Finally, the modified free boundary problem is stated in Section 3.4.

3.1. **The Second Order Operator for ρ.**

We recall the second order equation (2.5) for ρ, and we define the nonlinear operator

$$Q(\rho) := ((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta)_\xi + ((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi)_\eta + \xi\rho_\xi + \eta\rho_\eta.$$

We rewrite (2.5) in polar coordinates and obtain

$$\left((c^2(\rho) - r^2)\,\rho_r\right)_r + \frac{c^2(\rho)}{r}\rho_r + \left(\frac{c^2(\rho)}{r^2}\rho_\theta\right)_\theta = 0.$$

To ensure strict ellipticity of this equation, we introduce two cut-off functions

$$\phi_i(x) := \begin{cases} x, & x > \delta_i \\ \delta_i, & x \le \delta_i \end{cases} \qquad i \in \{1, 2\}, \tag{3.1}$$

for $\delta_1, \delta_2 > 0$ to be determined in terms of the fixed parameters k, ρ_0, ρ_1 and ϵ_*. The constant δ_1 will be chosen in Section 6 and the constant δ_2 will be specified in

(4.5). We modify each function ϕ_i so that it is smooth in a neighborhood of $x = \delta_i$ and that $\phi_1' \in [0,1]$ and $\phi_2' \geq 0$. We consider the modified equation

$$\left(\phi_1(c^2(\rho) - r^2)\,\rho_r\right)_r + \frac{c^2(\rho)}{r}\rho_r + \left(\frac{\phi_2(c^2(\rho))}{r^2}\,\rho_\theta\right)_\theta = 0. \tag{3.2}$$

We rewrite equation (3.2) in self-similar Cartesian coordinates to get $\tilde{Q}(\rho) = 0$ with

$$\begin{aligned}\tilde{Q}(\rho) :=& \frac{\phi_1\,\xi^2 + \phi_2\,\eta^2}{\xi^2+\eta^2}\rho_{\xi\xi} + 2\xi\eta\frac{\phi_1 - \phi_2}{\xi^2+\eta^2}\rho_{\xi\eta} + \frac{\phi_1\,\eta^2 + \phi_2\,\xi^2}{\xi^2+\eta^2}\rho_{\eta\eta} \\ &+ \left\{\frac{c^2 - \phi_2}{\xi^2+\eta^2} - 2\phi_1'\right\}\{\xi\rho_\xi + \eta\rho_\eta\} \\ &+ \frac{2cc'}{\xi^2+\eta^2}\left\{\phi_1'\,(\xi\rho_\xi + \eta\rho_\eta)^2 + \phi_2'\,(\eta\rho_\xi - \xi\rho_\eta)^2\right\},\end{aligned} \tag{3.3}$$

where the functions ϕ_1 and ϕ_1' are evaluated at $c^2(\rho) - (\xi^2+\eta^2)$, the functions ϕ_2 and ϕ_2' are evaluated at $c^2(\rho)$, while c and c' are evaluated at ρ. The eigenvalues of the operator $\tilde{Q}$ are

$$\lambda_1(\rho) = \phi_1\left(c^2(\rho) - (\xi^2+\eta^2)\right) \quad \text{and} \quad \lambda_2(\rho) = \phi_2\left(c^2(\rho)\right),$$

and $\tilde{Q}$ is strictly elliptic since

$$\lambda(\rho) := \min\{\lambda_1(\rho), \lambda_2(\rho)\} \geq \min\{\delta_1, \delta_2\} > 0. \tag{3.4}$$

3.2. **Oblique Derivative Boundary Condition.** As in [7], we write system (2.4) in conservation form

$$\partial_\xi\begin{pmatrix} m - \xi\rho \\ p - \xi m \\ -\xi n\end{pmatrix} + \partial_\eta\begin{pmatrix} n - \eta\rho \\ -\eta m \\ p - \eta n\end{pmatrix} = -2\begin{pmatrix}\rho \\ m \\ n\end{pmatrix}.$$

The Rankine-Hugoniot relations along the reflected shock

$$S\,:\,\xi(\eta),\ \eta \leq \eta_s, \tag{3.5}$$

separating states $U = (\rho, m, n)$ and $U_0 = (\rho_0, 0, n_0)$, are

$$[m] - \xi[\rho] = \frac{d\xi}{d\eta}([n] - \eta[\rho]), \quad [p] - \xi[m] = -\frac{d\xi}{d\eta}\eta[m], \quad -\xi[n] = \frac{d\xi}{d\eta}([p] - \eta[n]). \tag{3.6}$$

As in [7], we derive the condition

$$\beta\cdot\nabla\rho = 0 \quad \text{on} \quad \Sigma. \tag{3.7}$$

Here, $\nabla\rho := (\rho_\xi, \rho_\eta)$ and $\beta := (\beta_1, \beta_2)$ is given by

$$\begin{aligned}\beta_1 &= \xi'(\xi^2+\eta^2)(c^2(\rho)+s^2)(\xi - \eta\xi') - 2\xi\xi' s^2(c^2(\rho)+\eta^2) \\ &\quad -2s^2\eta(c^2(\rho)-\xi^2)(1-(\xi')^2) + 2\xi'\xi s^2(\xi^2 - c^2(\rho)), \\ \beta_2 &= (c^2(\rho)+s^2)(\xi^2+\eta^2)(\xi-\eta\xi') + 2s^2\xi'\eta(c^2(\rho)-\eta^2) \\ &\quad -2s^2\xi(c^2(\rho)-\eta^2)(1-(\xi')^2) + 2\xi'\eta s^2(c^2(\rho)+\xi^2),\end{aligned} \tag{3.8}$$

where s^2 is defined by (2.14). We define the operator

$$N(\rho) := \beta\cdot\nabla\rho. \tag{3.9}$$

The operations by which the equations (2.13) and (3.7) are derived from the Rankine-Hugoniot relations (2.12) and (3.6), respectively, can be reversed up to a constant.

Let

$$\nu := \frac{1}{1+(\xi')^2}(-1,\xi')$$

denote the inward unit normal to the curve (3.5) describing the reflected shock, and let us assume $\xi(\eta) \equiv r(\theta) \in \mathcal{K}$. We compute

$$\beta \cdot \nu = \frac{2s^2(\xi'\xi+\eta)}{1+(\xi')^2}\left\{(c^2(\rho)-\eta^2)(\xi')^2 + 2\xi\eta\xi' + c^2(\rho) - \xi^2\right\}.$$

We remark that $\xi'\xi + \eta = 0$ if and only if the curve $\xi(\eta)$ is tangent to a circle centered at the origin, which is ruled out by the monotonicity property (2.16) of the curves in the set $\mathcal{K}$. (Note that by symmetry, the reflected shock S is tangent to a circle at the point Ξ_0, Figure 4, and this is why we have to exclude Ξ_0 from the domain Ω.) Moreover, since the expression $\xi'\xi + \eta$ is positive at the reflection point Ξ_s, the uniform monotonicity property of the curves in $\mathcal{K}$ implies that there exists a constant C such that

$$\xi'\xi + \eta \geq C > 0 \tag{3.10}$$

holds uniformly in $\mathcal{K}$. Further, we introduce the polynomial

$$P(Y) := (c^2(\rho)-\eta^2)Y^2 + 2\xi\eta Y + c^2(\rho) - \xi^2, \tag{3.11}$$

and remark that if $P(\xi') > 0$, then $\beta\cdot\nu > 0$ and the operator N is oblique on Σ. Note that $P(\xi'(\eta_s)) > 0$ and that the discriminant of P is negative if $\xi^2+\eta^2 < c^2(\rho(\xi,\eta))$. Thus, $P(\xi') > 0$ holds at all points of the curve $\xi(\eta)$ where ρ is strictly subsonic. For the purpose of setting up an iteration, in which ρ may not always be subsonic at every point on the curve $\xi(\eta)$, we modify β by introducing a cut-off as follows.

We define a polynomial

$$G(Y) := \begin{cases} P(Y) = (c^2(\rho)-\eta^2)Y^2 + 2\xi\eta Y + c^2(\rho) - \xi^2, & \xi^2+\eta^2 < c^2(\rho) - \delta_1, \\ (\xi Y + \eta)^2 + \delta_1(Y^2+1), & \xi^2+\eta^2 \geq c^2(\rho) - \delta_1, \end{cases}$$

where δ_1 is a positive parameter as in (3.1). We introduce a modification of β in (3.8)

$$\chi = \begin{cases} (\beta_1,\beta_2), & \xi^2+\eta^2 < c^2(\rho) - \delta_1, \\ (\chi_1,\chi_2), & \xi^2+\eta^2 \geq c^2(\rho) - \delta_1, \end{cases} \tag{3.12}$$

in which c^2 is replaced by $\xi^2+\eta^2+\delta_1$ when $c^2(\rho) \leq \xi^2+\eta^2+\delta_1$, so

$$\begin{aligned} \chi_1 &= \xi'(\xi^2+\eta^2)(\xi^2+\eta^2+\delta_1+s^2)(\xi-\eta\xi') - 2\xi\xi's^2(\xi^2+2\eta^2+\delta_1) \\ &\quad -2s^2\eta(\eta^2+\delta_1)(1-(\xi')^2) - 2\xi'\xi s^2(\eta^2+\delta_1), \\ \chi_2 &= (\xi^2+\eta^2+\delta_1+s^2)(\xi^2+\eta^2)(\xi-\eta\xi') + 2s^2\xi'\eta(\xi^2+\delta_1) \\ &\quad -2s^2\xi(\xi^2+\delta_1)(1-(\xi')^2) + 2\xi'\eta s^2(2\xi^2+\eta^2+\delta_1) \end{aligned}$$

there. We define the operator

$$\tilde{N}(\rho) = \chi \cdot \nabla\rho. \tag{3.13}$$

Note that if $\xi^2+\eta^2 \geq c^2(\rho) - \delta_1$, then

$$\begin{aligned} \chi\cdot\nu &= \frac{2s^2(\xi'\xi+\eta)}{1+(\xi')^2}\left\{(\xi'\xi+\eta)^2 + \delta_1((\xi')^2+1)\right\} \\ &= \frac{2s^2(\xi'\xi+\eta)}{1+(\xi')^2}\,G(\xi') \;\geq\; \frac{2s^2(\xi'\xi+\eta)}{1+(\xi')^2}\,\delta_1 \;>\; 0. \end{aligned} \tag{3.14}$$

Hence, if the boundary Σ is described by a curve $\xi(\eta) \equiv r(\theta) \in \mathcal{K}$, then the operator $\tilde{N}$ is uniformly oblique on Σ.

3.3. **Shock Evolution Equation.** In order for the equation of the reflected shock (2.13) to be well-defined we replace it by

$$\frac{dr}{d\theta} = r\sqrt{\psi\left(\frac{r^2}{s^2} - 1\right)}. \tag{3.15}$$

Here

$$\psi(x) := \begin{cases} x, & x > \delta_* \\ \delta_*, & x \le \delta_*, \end{cases} \tag{3.16}$$

where δ_* is the same positive parameter as in (2.16) which will be specified in Section 6 in terms of the a priori fixed parameters ρ_0, ρ_1 and k. Since we will need ψ' to be continuous, we modify ψ so that it is smooth in a neighborhood of $x = \delta_*$.

3.4. **The Statement of the Modified Free Boundary Problem.** Our objective is to prove existence of a solution to the following modified problem.

Theorem 3.1. *(Modified free boundary problem)*
Let $\rho_0 > \rho_1 > 0$, $k \in (0, k_C(\rho_0, \rho_1))$, $\theta^* \in (-\pi/2, \pi/2)$, $\epsilon_* \in (0, \rho_s - \rho_0)$ and $\delta_1 > 0$ be given. There exist positive parameters δ_*, δ_2 and γ_0 such that for any $\gamma \in (0, \min\{1, \gamma_0\})$, $\alpha_{\mathcal{K}} = \gamma/2$ and any function $f \in H_\gamma$ satisfying (2.17), the free boundary problem for ρ and r given by

$$\begin{array}{l} \tilde{Q}(\rho) = 0 \quad \text{in} \quad \Omega, \\ \tilde{N}(\rho) = 0 \quad \text{on} \quad \Sigma, \\ r'(\theta) = r\sqrt{\psi\left(\frac{r^2}{s^2} - 1\right)} \quad \text{on} \quad \Sigma, \qquad r(\pi/2) = \eta_s, \\ \rho = f \quad \text{on} \quad \sigma, \qquad \rho_\xi = 0 \quad \text{on} \quad \Sigma_0, \qquad \rho(\Xi_s) = \rho_s, \end{array} \tag{3.17}$$

has a solution $\rho \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ in Ω and $r \in H_{1+\alpha_{\mathcal{K}}}$.

We break the proof of Theorem 3.1 into two steps.

Step 1 is to solve the fixed boundary value problem obtained by replacing the free boundary in Theorem 3.1 by a curve r chosen from the set $\mathcal{K}$. Again, assume we are given $\rho_0 > \rho_1 > 0$, $k \in (0, k_C(\rho_0, \rho_1))$, $\theta^* \in (-\pi/2, \pi/2)$, $\epsilon_* \in (0, \rho_s - \rho_0)$ and the positive parameters δ_1 and δ_*. We show that there exist $\delta_2 > 0$ and $\gamma_0 > 0$, depending only on ρ_0, ρ_1, k, θ^*, ϵ_*, δ_1 and δ_*, such that for any $\gamma \in (0, \min\{\gamma_0, 1\})$, $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$, a fixed curve $r \in \mathcal{K}$ defining Σ and a function $f \in H_\gamma$ satisfying (2.17), the nonlinear fixed boundary problem

$$\begin{array}{l} \tilde{Q}(\rho) = 0 \quad \text{in} \quad \Omega, \\ \tilde{N}(\rho) = 0 \quad \text{on} \quad \Sigma, \\ \rho = f \quad \text{on} \quad \sigma, \qquad \rho_\xi = 0 \quad \text{on} \quad \Sigma_0, \qquad \rho(\Xi_s) = \rho_s, \end{array} \tag{3.18}$$

has a solution $\rho \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ in the domain Ω.

Step 2 is to define a mapping using the shock evolution equation. We update the position of the reflected shock using the initial value problem

$$\begin{cases} \tilde{r}'(\theta) = \tilde{r}(\theta)\sqrt{\psi\left(\frac{\tilde{r}(\theta)^2}{s^2(\rho(r(\theta),\theta),\rho_0)} - 1\right)}, & \theta \in (\theta^*, \pi/2), \\ \tilde{r}\,(\pi/2) = \eta_s. \end{cases} \tag{3.19}$$

This defines a map $J : r \mapsto \tilde{r}$ on the set $\mathcal{K}$. We show that we can choose δ_* in terms of ρ_0, ρ_1 and k, so that there exists $\gamma_0 > 0$ (possibly smaller than γ_0 found in the previous step), also depending on the a priori fixed parameters ρ_0, ρ_1, k, θ^*, ϵ_* and δ_1, such that for any $\gamma \in (0, \min\{1, \gamma_0\})$ and $\alpha_{\mathcal{K}} = \gamma/2$, the map J has a fixed point $r \in \mathcal{K}$. With this fixed point $r(\theta)$, $\theta \in (\theta^*, \pi/2)$, defining the boundary $\Sigma = \{(r(\theta), \theta) : \theta \in (\theta^*, \pi/2)\}$, the corresponding solution $\rho \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ to the fixed boundary problem (3.18) solves the modified free boundary problem (3.17).

The first step is completed in Section 4 and the second in Section 5.

4. Solution to the Modified Fixed Boundary Problem

In this section we find positive parameters δ_2 and γ_0, depending only on ρ_0, ρ_1, k, θ^*, ϵ_*, δ_1 and δ_*, such that for any $\gamma \in (0, \min\{\gamma_0, 1\})$, $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$, a fixed $r \in \mathcal{K}$ describing the boundary Σ and a function $f \in H_\gamma$ satisfying (2.17), the fixed boundary problem (3.18) has a solution $\rho \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ in Ω. We use the result in Section 4 of [15] which applies to fixed nonlinear boundary problems of the second order where the operators in the domain and on the boundary satisfy certain structural conditions. These conditions are stated in Section 4.3 in [15] and, for convenience, we give them in Appendix B using the notation of this paper. We confirm in Proposition 4.1 that they hold for the problem (3.18), arising from transonic regular reflection for the NLWS, and the result follows from Theorem 4.7 in [15].

Proposition 4.1. *For any curve $r \in \mathcal{K}$ fixed, the boundary value problem* (3.18) *satisfies the structural conditions* (6.15)-(6.20). *Moreover, for*

$$K \ge \max\left\{\frac{2c(\rho_s)c'(\rho_s)}{\delta_1}, 4(c'(\rho_s))^2\right\} \tag{4.1}$$

the inequality (6.21) *holds.*

Proof. First, we write the operator $\tilde{Q}$, given by (3.3), as in (6.14). We note that for a fixed curve $r \in \mathcal{K}$, the coefficients a_{ij}, b_i and c_{ij} are in C^1 and that the coefficients χ_i of the vector χ, given by (3.12) and defining the operator $\tilde{N}$ in (3.13) are such that $\chi_i \in C^2$.

Recall that the operators $\tilde{Q}$ and $\tilde{N}$, given by (3.3) and (3.13), are strictly elliptic in Ω and oblique on Σ, respectively, by (3.4) and (3.14). Clearly, the operator $\rho_\xi = (1, 0) \cdot \nabla\rho$ is both strictly and uniformly oblique on Σ_0. Hence, the conditions of Lemma 4.8 in [15] are satisfied. By this lemma, given $r \in \mathcal{K}$, describing the boundary Σ, and a solution $\rho \in C^1(\Omega)$ to the fixed problem (3.18) we have uniform L^∞ bounds

$$\rho_0 + \epsilon_* \le \rho(\xi, \eta) \le \rho_s, \quad (\xi, \eta) \in \Omega. \tag{4.2}$$

Next, we show that the uniform bounds (4.2) imply uniform ellipticity of the operator $\tilde{Q}$ and both strict and uniform obliqueness of the operator $\tilde{N}$. The operator $\tilde{Q}$ is uniformly elliptic in Ω since

$$\frac{\Lambda(\rho)}{\lambda(\rho)} := \frac{\max\{\lambda_1(\rho), \lambda_2(\rho)\}}{\min\{\lambda_1(\rho), \lambda_2(\rho)\}} \le \frac{c^2(\rho_s)}{\min\{\delta_1, \delta_2\}}. \tag{4.3}$$

Further, recall the definition (3.12) of the vector χ and note that from (3.14), the uniform bounds (4.2) on ρ and the uniform bounds on $r \in \mathcal{K}$ we have

$$\chi \cdot \nu \geq C > 0, \qquad \text{for all } \rho \text{ and } r \in \mathcal{K},$$

for some constant C. Therefore, the operator $\tilde{N}$ is strictly oblique on Σ. Moreover, we have a uniform bound

$$|\chi| = \sqrt{\chi_1^2 + \chi_2^2} \leq C,$$

again using the uniform bounds on the curve $\xi(\eta) \equiv r(\theta) \in \mathcal{K}$ describing the boundary Σ and the uniform bound (4.2) on the solution ρ. Therefore

$$\frac{\chi \cdot \nu}{|\chi|} = \frac{2\, s^2\, (\xi'\xi + \eta)\, G(\xi')}{|\chi|} \geq \frac{2\, c^2(\rho_0 + \epsilon_*)\, c\, \delta_1}{C} > 0, \tag{4.4}$$

where c is the constant in (3.10). Hence, the operator $\tilde{N}$ is also uniformly oblique on Σ. This confirms that conditions (6.15)-(6.18) hold.

Note that by choosing δ_2, in the definition (3.1) of the cut-off function ϕ_2, such that

$$0 < \delta_2 \leq c^2(\rho_0 + \epsilon_*), \tag{4.5}$$

the function ϕ_2 is equal to the identity function. We assume the choice (4.5) for δ_2. Therefore, the operator $\tilde{Q}$ in (3.3) becomes

$$\begin{aligned}\tilde{Q}(\rho) =& \frac{\phi_1\, \xi^2 + c^2\, \eta^2}{\xi^2 + \eta^2} \rho_{\xi\xi} + 2\xi\eta \frac{\phi_1 - c^2}{\xi^2 + \eta^2} \rho_{\xi\eta} + \frac{\phi_1\, \eta^2 + c^2\, \xi^2}{\xi^2 + \eta^2} \rho_{\eta\eta} \\ & - 2\phi_1' \left\{\xi\rho_\xi + \eta\rho_\eta\right\} + \frac{2cc'}{\xi^2 + \eta^2} \left\{\phi_1' \left(\xi\rho_\xi + \eta\rho_\eta\right)^2 + 2cc' \left(\eta\rho_\xi - \xi\rho_\eta\right)^2\right\}, \\ =& \sum_{i,j} a_{ij}(\rho, \xi, \eta) D^{ij}\rho + \sum_i b_i(\rho, \xi, \eta) D^i\rho + \sum_{i,j} c_{ij}(\rho, \xi, \eta) D^i\rho D^j\rho.\end{aligned} \tag{4.6}$$

Here, the functions c and c' are evaluated at ρ, and ϕ_1 and ϕ_1' are evaluated at $c^2(\rho) - (\xi^2 + \eta^2)$.

Clearly, the condition (6.19) holds for the operator $\tilde{Q}$ given by (4.6) and next we check that (6.20) is also satisfied. We have

$$\begin{aligned}|\sum_{i,j} a_{ij}(\rho, \xi, \eta) D^{ij}\rho| &\leq (\eta_s + 2cc' + 4c^2(c')^2)(|\rho_\xi|^2 + |\rho_\eta|^2) + 2\eta_s \\ &\leq \min\{\delta_1, \delta_2\} \left(\frac{\eta_s + 2cc' + 4c^2(c')^2}{\min\{\delta_1, \delta_2\}} \sum_i |D^i\rho|^2 + \frac{2\eta_s}{\min\{\delta_1, \delta_2\}}\right).\end{aligned}$$

Hence, (6.20) holds with

$$\mu_0 = \frac{\eta_s + 2c(\rho_s)c'(\rho_s) + 4c^2(\rho_s)(c'(\rho_s))^2}{\min\{\delta_1, \delta_2\}} \quad \text{and} \quad \Phi = \frac{2\eta_s}{\min\{\delta_1, \delta_2\}}. \tag{4.7}$$

Finally, we check that (6.21) holds for the parameter K as in (4.1). Let $r \in \mathcal{K}$ be arbitrary and let ρ be a solution to the equation $\tilde{Q}(\rho) = 0$. It is easy to show

$$\begin{aligned}&K \sum_{i,j} a_{ij}(\rho, \xi, \eta) D^i\rho D^j\rho - \sum_{i,j} c_{ij}(\rho, \xi, \eta) D^i\rho D^j\rho = \\ &\qquad \frac{1}{\xi^2 + \eta^2} \left\{(K\phi_1 - 2cc'\phi_1')(\xi\rho_\xi + \eta\rho_\eta)^2 + (Kc^2 - 4c^2(c')^2)(\eta\rho_\xi - \xi\rho_\eta)^2\right\}.\end{aligned}$$

Hence, (6.21) holds. $\square$

Therefore, the structural conditions of Theorem 4.7 in [15] are satisfied. By this theorem, there exists $\gamma_0 > 0$, depending on the sizes of the opening angles of the domain Ω at the set of corners $\mathbf{V}$ and on the bounds on the ellipticity ratio of the operator $\tilde{Q}$, such that for every $\gamma \in (0, \min\{\gamma_0, 1\})$, $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$, $r \in \mathcal{K}$ and any function $f \in H_\gamma$ satisfying (2.17), there exists a solution ρ to the fixed boundary problem (3.18). Also, we have $\rho \in H^{(-\gamma)}_{1+\alpha_*}$, for all $\alpha_* \in (0, \alpha_{\mathcal{K}}]$.

Remark 4.2. We note that by the definition of the set $\mathcal{K}$ of admissible curves, the sizes of the opening angles of the domain Ω at the set of corners $\mathbf{V}$ satisfy bounds depending only on the parameters ρ_0, ρ_1, k and θ^*, which are fixed throughout the paper, and on the parameter δ_* which will be chosen in Section 5 also in terms of ρ_0, ρ_1, k and θ^*. Therefore, the parameter γ_0, given by Theorem 4.7 in [15], can be taken independent of the choice of the curve $r \in \mathcal{K}$. Moreover, using the uniform bounds (4.3) on the ellipticity ratio of the operator $\tilde{Q}$ and the choice of δ_2 in (4.5), we have that γ_0 depends only on the fixed parameters ρ_0, ρ_1, k, θ^* and ϵ_*, and the parameters δ_1 and δ_* which will be chosen in Section 5 and Section 6, respectively, also in terms of ρ_0, ρ_1, k and ϵ_*.

5. Solution to the Modified Free Boundary Problem

In this section we complete the second step of the proof of Theorem 3.1.

Let $\gamma_0 > 0$ be the parameter found in Section 4. Let $\gamma \in (0, \min\{\gamma_0, 1\})$ and let $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$ be arbitrary. For any $r \in \mathcal{K}$, describing the boundary Σ, and any function $f \in H_\gamma$ satisfying (2.17), we find a solution $\rho(\xi, \eta)$ to the nonlinear fixed boundary problem (3.18). We define the curve $\tilde{r}(\theta)$, $\theta \in (\theta^*, \pi/2)$, as a solution to (3.19). This gives a map $J : \rho \mapsto \tilde{\rho}$ on the set $\mathcal{K}$. We show that J has a fixed point using the following

Theorem. (Corollary 11.2 in [14]) Let $\mathcal{K}$ be a closed and convex subset of a Banach space $\mathcal{B}$ and let $J : \mathcal{K} \to \mathcal{K}$ be a continuous mapping so that $J(\mathcal{K})$ is precompact. Then J has a fixed point.

We take $\mathcal{B}$ to be the space $H_{1+\alpha_{\mathcal{K}}}$, and we take $\mathcal{K}$ as in Section 2.3. In this section we specify the parameter δ_* in the definition of the set $\mathcal{K}$ and the cut-off function ψ (see (3.16)), and we further specify γ and $\alpha_{\mathcal{K}}$ so that the hypotheses of the previous fixed point theorem are satisfied.

Lemma 5.1. *Let the parameters $\rho_0 > \rho_1 > 0$, $k \in (0, k_C(\rho_0, \rho_1))$, $\theta^* \in (-\pi/2, \pi/2)$ and $\epsilon_* \in (0, \rho_s - \rho_0)$ be given. Let δ_* be such that*

$$0 < \delta_* < \frac{\eta_s^2}{s^2(\rho_s, \rho_0)} - 1. \tag{5.1}$$

There exists $\gamma_0 > 0$ such that for any $\gamma \in (0, \min\{1, \gamma_0\})$ and $\alpha_{\mathcal{K}} = \gamma/2$, we have

(a) $J(\mathcal{K}) \subseteq \mathcal{K}$, and

(b) the set $J(\mathcal{K})$ is precompact in $H_{1+\alpha_{\mathcal{K}}}$.

Remark 5.2. Recall from (2.10) that we have $\rho_s > \rho_0$, implying, by the monotonicity of the function $s^2(\cdot, \rho_0)$, that $s^2(\rho_s, \rho_0) > c^2(\rho_0)$. Note that the choice of δ_* in (5.1) gives that

$$\delta_* < e^{(\pi - 2\theta^*)\sqrt{\eta_s^2/c^2(\rho_0) - 1}} \left(\frac{\eta_s^2}{c^2(\rho_0)} - 1 \right),$$

and, in particular, the monotonicity condition (2.16) in the definition of the set $\mathcal{K}$ makes sense.

Proof. (of Lemma 5.1) This proof follows ideas from Section 4.2.1 in [4] and some of its parts are identical to the proof of Lemma 5.3 in [15].

Let γ_0 be the parameter found in Section 4. Let $\gamma \in (0, \min\{\gamma_0, 1\})$ be arbitrary and let $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$. Let $r \in \mathcal{K}$ and $f \in H_\gamma$ satisfying (2.17) be given, and let $\rho(\xi, \eta) \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ be a solution to the fixed boundary problem (3.18) found in Section 4. Further, suppose that $\tilde{r}(\theta)$, $\theta \in (\theta^*, \pi/2)$, is a solution to the problem (3.19).

To show (a) we need to show that $\tilde{\rho} \in \mathcal{K}$. Clearly, $\tilde{r}(\pi/2) = \eta_s$, and

$$\tilde{r}'(\pi/2) = \eta_s \sqrt{\psi\left(\frac{\eta_s^2}{s^2(\rho_s, \rho_0)} - 1\right)} = \eta_s \sqrt{\frac{\eta_s^2}{s^2(\rho_s, \rho_0)} - 1},$$

by the choice of δ_*. Next, note $\tilde{r}'(\theta) \geq \tilde{r}(\theta)\sqrt{\delta_*}$, implying $\frac{d\tilde{r}}{\tilde{r}} \geq \sqrt{\delta_*}$. After integrating from θ to $\pi/2$, we get

$$\tilde{r}(\theta) \leq \eta_s, \quad \theta \in (\theta^*, \pi/2). \tag{5.2}$$

On the other hand,

$$\begin{aligned} \tilde{r}'(\theta) &\leq \tilde{r}(\theta) \sqrt{\psi\left(\frac{\eta_s^2}{c^2(\rho_0)} - 1\right)} \\ &\leq \tilde{r}(\theta) \sqrt{\frac{\eta_s^2}{c^2(\rho_0)} - 1}, \quad \text{by the choice of } \delta_*, \end{aligned}$$

implying $\frac{d\tilde{r}}{\tilde{r}} \leq \sqrt{\frac{\eta_s^2}{c^2(\rho_0)} - 1}$, and after integrating from θ to $\pi/2$ we obtain

$$\tilde{r}(\theta) \geq \frac{\eta_s}{e^{(\pi/2-\theta)\sqrt{\eta_s^2/c^2(\rho_0)-1}}} \geq \frac{\eta_s}{e^{(\pi/2-\theta^*)\sqrt{\eta_s^2/c^2(\rho_0)-1}}}.$$

Together with (5.2), this implies the desired boundedness of the curve $\tilde{r}(\theta)$. Once this boundedness is established, the required monotonicity is clear.

It is left to show that we can find γ and $\alpha_{\mathcal{K}}$ so that

$$\tilde{r} \in H_{1+\alpha_{\mathcal{K}}} \tag{5.3}$$

and that (b) holds. This part of the proof is identical to the proof of Lemma 5.3 in [15]. In short, Theorem 2.3 in [21] gives that there exist α_0 and C such that a solution ρ to the fixed boundary problem (3.18) satisfies

$$[\rho]_{\alpha_0} \leq C$$

in a neighborhood of Σ. Here, α_0 depends on the bounds for the ellipticity ratio of the operator $\tilde{Q}$ and on the obliqueness constant of the operator $\tilde{N}$, and on $\mu_0|\rho|_0$, where μ_0 is the constant in (4.7). The constant C also depends on Ω. Using the bound (4.3) for the ellipticity ratio of $\tilde{Q}$ and the choice (4.5) for δ_2, the bound (4.4) for the obliqueness constant of the operator $\tilde{N}$, uniform bounds (4.2) on the solution ρ, the definition of the set $\mathcal{K}$ and the choice (5.1) for δ_*, we have that α_0 and C depend only on the fixed parameters ρ_0, ρ_1, k, θ^* and ϵ_*, and the parameter δ_1 which will be chosen in Section 6 also in terms of ρ_0, ρ_1, k and ϵ_*. We replace γ_0 by $\min\{\gamma_0, \alpha_0\}$ and we take $\gamma \in (0, \min\{1, \gamma_0\})$. This implies $|\tilde{r}'|_\gamma \leq C$ and

$$|\tilde{r}|_{1+\gamma} \leq C(\pi/2 - \theta^*). \tag{5.4}$$

Therefore, $\tilde{r} \in H_{1+\gamma}$. We choose $\alpha_{\mathcal{K}} \in (0, \gamma]$ to ensure (5.3). Since (5.4) holds independently of $\tilde{r}$, we have that the set $J(\mathcal{K})$ is contained in a bounded set in $H_{1+\gamma}$ and to show (b) we take $\alpha_{\mathcal{K}} = \gamma/2$. □

We also note that the map $J : \mathcal{K} \to \mathcal{K}$ is continuous. Therefore, the hypothesis of the fixed point theorem from the beginning of this section (Corollary 11.2 in [14]) are satisfied and the map J has a fixed point $r \in \mathcal{K}$. We use this curve $r(\theta)$, $\theta \in (\theta^*, \pi/2)$, to specify the boundary Σ, and using Section 4 we find a solution $\rho \in H^{(-\gamma)}_{1+\alpha_*}$, for all $\alpha_* \in (0, \alpha_{\mathcal{K}}]$, of the modified free boundary problem (3.17).

Remark 5.3. Once the density component ρ is determined in the domain Ω, we find the momenta m and n in Ω from the second and the third equations in (2.4). These two equations are the transport equations for m and n:

$$\frac{\partial m}{\partial s} = p_\xi \quad \text{and} \quad \frac{\partial n}{\partial s} = p_\eta, \tag{5.5}$$

where $s = (\xi^2 + \eta^2)/2$ stands for the radial variable. Note that m and n are known in the hyperbolic part of the domain and along the boundary Σ using the Rankine-Hugoniot relations (3.6). We find m and n in the domain Ω by integrating the equations (5.5) from Σ towards the origin. We note that $\nabla p \in H_\alpha$ and, hence, ∇p is absolutely integrable on Σ.

6. Proof of Theorem 2.3

In this section we discuss the conditions under which ρ, a solution to the modified free boundary problem in Theorem 3.1, together with m and n as in Remark 5.3, solves the free boundary problem in Theorem 2.3. More precisely, we investigate when the cut-off functions ϕ_1, ϕ_2, χ and ψ can be removed. Recall that the functions ϕ_1 and ϕ_2 are introduced in (3.1) so that the operator $\tilde{Q}$ given by (3.3) is strictly elliptic, the function χ is given by (3.12) and ensures that the operator $\tilde{N}$ defined in (3.13) is oblique and ψ, given by (3.16), is introduced so that the equation (3.15) of the evolution of the reflected shock is well-defined.

Recall that we choose the parameter δ_2 in the definition of ϕ_2 so that the bounds (4.5) hold. This implies that the cut-off function ϕ_2 is identity.

Next we show that in a neighborhood of the reflection point $\Xi_s = (0, \eta_s)$ the cut-off functions ϕ_1 and ψ can be replaced by identity and the cut-off function χ can be replaced by β. Note that at Ξ_s we have

$$c^2(\rho) - (\xi^2 + \eta^2) = c^2(\rho_s) - \eta_s^2 > 0,$$

because of our assumption that the point Ξ_s is subsonic with respect to the state $U_s = (\rho_s, m_s, n_s)$ (see (2.11)). Further, note that at Ξ_s we have

$$\frac{r^2}{s^2(\rho, \rho_0)} - 1 = \frac{\eta_s^2}{s^2(\rho_s, \rho_0)} - 1 > 0 \tag{6.1}$$

by Remark 2.1. Since the functions

$$c^2(\rho) - (\xi^2 + \eta^2) \quad \text{and} \quad \frac{\xi^2 + \eta^2}{s^2(\rho, \rho_0)} - 1, \quad (\xi, \eta) \in \Omega,$$

are positive at the reflection point Ξ_s, by continuity we have that these two functions are positive in a closed neighborhood $\bar{\mathcal{N}}$ of Ξ_s. We take the parameters δ_1 and δ_*

such that

$$\delta_1, \delta_* \in \left(0, \min_{(\xi,\eta)\in\bar{\mathcal{N}}} \left\{ c^2(\rho) - (\xi^2+\eta^2), \frac{\xi^2+\eta^2}{s^2(\rho,\rho_0)} - 1 \right\}\right).$$

Hence, we can remove the cut-off functions ϕ_1, ψ and χ in the neighborhood $\mathcal{N}$ of the reflection point Ξ_s. Therefore, a solution ρ of the modified free boundary problem in Theorem 3.1, with m and n found as in Remark 5.3, solves the free boundary problem in Theorem 2.3 in the neighborhood $\mathcal{N}$.

Appendix A: Parameter Values for Regular Reflection

Consider the Riemann initial data (2.2) consisting of two sectors with states $U_0 = (\rho_0, 0, n_0)$ and $U_1 = (\rho_1, 0, 0)$, separated by half-lines $x = \pm ky$, $y \geq 0$, with k positive, as in Figure 1. We choose $\rho_0 > \rho_1 > 0$ arbitrary and we take

$$n_0 = \frac{\sqrt{1+k^2}}{k}\sqrt{(p(\rho_0) - p(\rho_1))(\rho_0 - \rho_1)}.$$

This implies that each of the two initial discontinuities $x = \pm ky$, $y \geq 0$, results in a one-dimensional solution consisting of a shock and a linear wave (Figure 2). In this part of the paper we describe how to choose the parameter k, depending on the densities ρ_0 and ρ_1, so that the above Riemann data leads to a transonic regular reflection.

Remark 6.1. Most of our discussion will be for a general function of pressure $p(\rho)$, $\rho > 0$, with property that

$$c^2(\rho) := p'(\rho),\ \rho > 0, \quad \text{is a positive and increasing function.} \tag{6.2}$$

We will give more details for the example of the γ-law pressure with $\gamma = 2$. We recall that a γ-law pressure relation is given by

$$p(\rho) = \rho^\gamma/\gamma, \quad \rho > 0,$$

for some $\gamma > 1$. We have $c^2(\rho) = \rho^{\gamma-1}$, $\rho > 0$, and we note that the system (2.1) admits a scaling

$$(x,y) \mapsto \rho_1^{(\gamma-1)/2}(x',y'), \quad \rho \mapsto \rho_1\rho' \quad \text{and} \quad (m,n) \mapsto \rho_1^{(\gamma+1)/2}(m',n').$$

Hence, in this case, the flow behavior depends only on the density ratio ρ_0/ρ_1, or, equivalently, on the velocity ratio or Mach number

$$M = \frac{c(\rho_0)}{c(\rho_1)} = \left(\frac{\rho_0}{\rho_1}\right)^{(\gamma-1)/2}. \tag{6.3}$$

Therefore, the Riemann data (2.2) can be parameterized in terms of ρ_0/ρ_1 and k.

Following the notation in Section 2.1, the one-dimensional Riemann solution with states U_0 and U_1, on the left and on the right, respectively, consists of the linear wave $l_a : \xi = k\eta$ connecting U_0 to the intermediate state $U_a = (\rho_0, m_a, n_a)$ and the shock $S_a : \xi = k\eta + \chi_a$ connecting U_a to U_1. Further, the one-dimensional solution with states U_1 and U_0, on the left and on the right, respectively consists of the linear wave $l_b : \xi = -k\eta$, the intermediate state $U_b = (\rho_0, -m_a, n_a)$ and the shock $S_b : \xi = -k\eta - \chi_a$ (see Figure 2). Here, χ_a, m_a and n_a are found using the Rankine-Hugoniot relations and are given by (2.7).

Let $\Xi_s = (0, \eta_s)$ denote the position of the projected intersection point of the shocks S_a and S_b. Recall, that η_s is given by (2.8). We distinguish the following three regions according to the position of the point Ξ_s:

- Region A: This region corresponds to those values of k, depending on ρ_0 and ρ_1, for which we have

$$\eta_s < c(\rho_0). \tag{6.4}$$

  Hence, the point Ξ_s is inside the sonic circle $C_0 : \xi^2 + \eta^2 = c^2(\rho_0)$. The two shocks S_a and S_b interact with C_0 and a regular reflection cannot happen.
- Region B: In this region the parameter $k(\rho_0, \rho_1)$ is specified so that we have

$$c(\rho_0) < \eta_s < \eta_*,$$

  where η_* is the value below which the quasi-one-dimensional problem at Ξ_s with states U_a and U_b on the left and on the right, respectively, does not have a solution. Therefore, in this case, the shocks S_a and S_b could intersect at the point Ξ_s, which is hyperbolic with respect to both states U_a and U_b. However, a regular reflection cannot occur because the quasi-one-dimensional problem at Ξ_s does not have a solution. We do not have scenario for the solution in this region.
- Region C: The value of the parameter $k(\rho_0, \rho_1)$ is such that

$$\eta_s > \eta_*. \tag{6.5}$$

  In other words, the shocks S_a and S_b intersect at the η-axis at the point Ξ_s and moreover, the quasi-one-dimensional Riemann problem at Ξ_s has a solution. Hence, a regular reflection occurs. We show in this section that there are, in general, two solutions to this quasi-one-dimensional Riemann problem, each consisting of two shocks. As in Section 2.1, we denote the two intermediate states for these two solutions by

$$U_R = (\rho_R, m_R, n_R) \quad \text{and} \quad U_F = (\rho_F, m_F, n_F), \tag{6.6}$$

  where we assume that $\rho_R \leq \rho_F$. We will further discuss for which values of k satisfying (6.5) we have a transonic regular reflection and we will explain the definition (2.11) of the state $U_s := U(\Xi_s)$.

The main goal of this section is to find the boundaries between the regions A, B and C. Following Remark 6.1, in the case of a γ-law pressure, these boundaries can be described by the curves in the $(\rho_0/\rho_1, k)$-plane. For $\gamma = 2$ they are numerically computed in Figure 5 (the region A being above the curve k_A, the region B is between k_A and k_C, and the region C is below the curve k_C).

Let us first consider the region A. Using the expression (2.8) for η_s, we get that the condition (6.4) is equivalent to

$$k > \frac{s(\rho_0, \rho_1)}{\sqrt{c^2(\rho_0) - s^2(\rho_0, \rho_1)}} =: k_A(\rho_0, \rho_1),$$

where $s(\cdot, \cdot)$ is defined in Lemma 2.2. Note that for a fixed $\rho_1 > 0$ we have

$$\lim_{\rho_0 \to \rho_1} k_A(\rho_0, \rho_1) = \infty$$

and

$$\lim_{\rho_0 \to \infty} k_A(\rho_0, \rho_1) = \lim_{\rho_0 \to \infty} \sqrt{\frac{p(\rho_0)}{\rho_0 c^2(\rho_0) - p(\rho_0)}}.$$

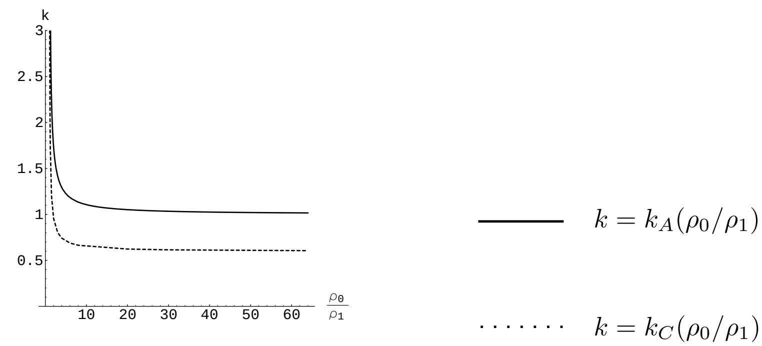

FIGURE 5. The curves k_A and k_C in the $(\rho_0/\rho_1, k)$-plane for the γ-law pressure, $\gamma = 2$.

For a γ-law pressure we can write

$$k_A\left(\frac{\rho_0}{\rho_1}\right) = \sqrt{\frac{F(M)}{\gamma M^2 - F(M)}}, \qquad \text{where} \qquad F(M) = \left(\frac{M^{\frac{2\gamma}{\gamma-1}} - 1}{M^{\frac{2}{\gamma-1}} - 1}\right),$$

and M is given by (6.3), and also

$$\lim_{\rho_0\to\infty} k_A\left(\frac{\rho_0}{\rho_1}\right) = \frac{1}{\sqrt{\gamma-1}}.$$

In the case $\gamma = 2$, we have (see Figure 5)

$$k_A\left(\frac{\rho_0}{\rho_1}\right) = \sqrt{\frac{\rho_0/\rho_1 + 1}{\rho_0/\rho_1 - 1}}.$$

Next we investigate the regions B and C, i.e., we suppose $\eta_s > c(\rho_0)$. Therefore, the projections of the shocks S_a and S_b intersect at the point Ξ_s, hyperbolic with respect to the states $U_a = (\rho_0, m_a, n_a)$ and $U_b = (\rho_0, -m_a, n_a)$, with values of m_a and n_a given in (2.7). We want to solve the quasi-one-dimensional Riemann problem at Ξ_s, along a line segment parallel to the ξ-axis, with states U_b and U_a on the left and on the right, respectively. (A general discussion on quasi-one-dimensional Riemann problems is given in [3] and formulas for a solution in the case of the NLWS are given in [5].) The condition for a solution to this quasi-one-dimensional Riemann problem to exist is that the shock loci $S^+(U_a)$ and $S^-(U_b)$ intersect. The formulas for $m(\rho)$ along the shock loci $S^\pm(U)$, for a given state U, are obtained in [5] (Appendix 6B). We have that if $U = (\rho, m, n) \in S^+(U_a)$, then

$$m(\rho) = m_a + \frac{p(\rho) - p(\rho_0)}{\eta_s}\sqrt{\frac{\eta_s^2(\rho - \rho_0)}{p(\rho) - p(\rho_0)} - 1}, \tag{6.7}$$

and if $U = (\rho, m, n) \in S^-(U_b)$, then

$$m(\rho) = -m_a - \frac{p(\rho) - p(\rho_0)}{\eta_s}\sqrt{\frac{\eta_s^2(\rho - \rho_0)}{p(\rho) - p(\rho_0)} - 1}.$$

Moreover, along both shock polars $S^+(U_a)$ and $S^-(U_b)$ we have

$$n(\rho) = n_a + \frac{p(\rho) - p(\rho_0)}{\eta_s}. \tag{6.8}$$

Note that (6.8) implies that the intersections of the projected shock loci $S^+(U_a)$ and $S^-(U_b)$ in the (ρ, m)-plane correspond to intersections of the loci in the (ρ, m, n)-space.

In Figures 6 and 7, we consider the γ-law pressure with $\gamma = 2$ and, for an example, we take $\rho_0 = 64$ and $\rho_1 = 1$. For the case of $k = 0.5$, we plot the projected shock loci $S^\pm(U_a)$ and $S^\pm(U_b)$ in the (ρ, m)-plane in Figure 6. In Figure 7, we vary the parameter k and depict the projections of the corresponding shock loci $S^+(U_a)$ and $S^-(U_b)$.

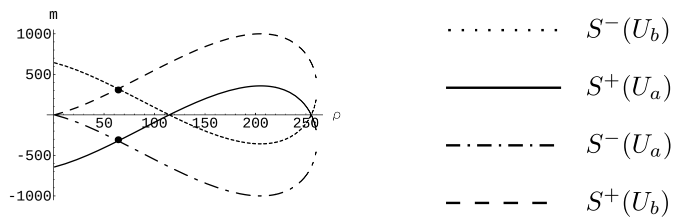

FIGURE 6. The projected shock loci for the states U_a and U_b in the (ρ, m)-plane.

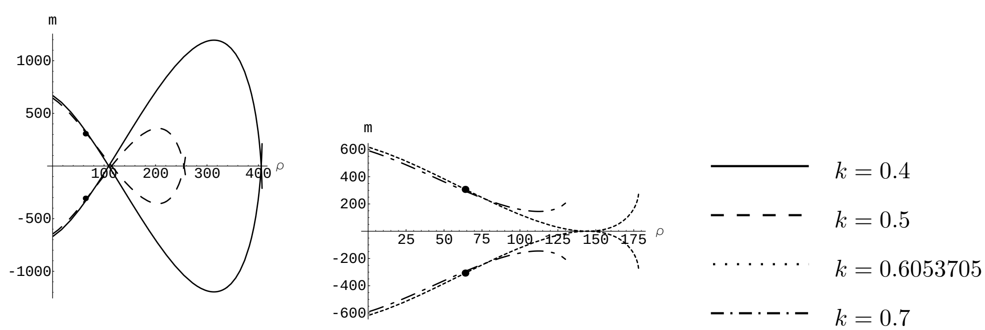

FIGURE 7. The projected shock loci $S^+(U_a)$ and $S^-(U_b)$ in the (ρ, m)-plane for different values of k.

Note that because of the symmetry of the states U_a and U_b, the boundary between the regions B and C occurs at those values of the parameter $k = k_C(\rho_0, \rho_1)$ for which

$$\max_{(\rho,m,n)\in S^+(U_a)} m(\rho) = 0.$$

To find the values of $k_C(\rho_0, \rho_1)$ at which the maximum of the function $m(\rho)$ given by (6.7) is zero, we solve the system of equations $m'(\rho) = 0$ and $m(\rho) = 0$, i.e.,

$$\begin{aligned} c^2(\rho)(\rho - \rho_0) + p(\rho) - p(\rho_0) - \tfrac{2(p(\rho)-p(\rho_0))c^2(\rho)}{\eta_s^2} &= 0, \\ m_a^2 = (p(\rho) - p(\rho_0))(\rho - \rho_0) - \left(\tfrac{p(\rho)-p(\rho_0)}{\eta_s}\right)^2&. \end{aligned} \tag{6.9}$$

We express ρ from the first equation in (6.9) and substitute into the second equation to find $k_C(\rho_0, \rho_1)$. Even in the case of the γ-law pressure with $\gamma = 2$, we obtain only an implicit relation between k_C and ρ_0/ρ_1, and we depict $k = k_C(\rho_0/\rho_1)$ numerically in Figure 5.

When $k = k_C(\rho_0, \rho_1)$, the two loci $S^+(U_a)$ and $S^-(U_b)$ are tangent, and they intersect at a single point. This implies that there exists a unique solution to the above quasi-one-dimensional Riemann problem at Ξ_s. When k is such that $0 < k < k_C(\rho_0, \rho_1)$, there are two points of intersection,

$$U_R = (\rho_R, m_R, n_R) \quad \text{and} \quad U_F = (\rho_F, m_F, n_F),$$

corresponding to different intermediate states in the two solutions of the quasi-one-dimensional Riemann problem at the reflection point Ξ_s. Note that, by the geometry of the shock loci $S^+(U_a)$ and $S^-(U_b)$ (see Figure 6) and by symmetry, we have

$$\rho_R, \rho_F > \rho_0 \quad \text{and} \quad m_R = m_F = 0.$$

We assume $\rho_F > \rho_R$. Each of the two solutions of the quasi-one-dimensional Riemann problem consists of a shock connecting the state U_b to an intermediate state (either U_R or U_F) and a shock connecting this intermediate state to U_a.

For the case of the γ-law pressure with $\gamma = 2$, we find numerically that $c(\rho_F) = \sqrt{\rho_F} > \eta_s$ for any $0 < k < k_C$, and that $c(\rho_R) = \sqrt{\rho_R} > \eta_s$, for sufficiently large values of k. More precisely, the point Ξ_s is within the sonic circle

$$C_R : \xi^2 + \eta^2 = c^2(\rho_R)$$

only if $k_* < k < k_C$, for some value $k_*(\rho_0/\rho_1)$. The curve $k = k_*(\rho_0/\rho_1)$ is depicted in Figure 8. Hence, the reflection point Ξ_s is subsonic for the state U_F for any $0 < k < k_C$ and is subsonic for the state U_R if $k_* < k < k_C$, explaining our definition of $U_s := U(\Xi_s)$ in (2.11). Even though we show this numerically for the γ-law pressure with $\gamma = 2$ (we checked it also for $\gamma = 3$), we believe that it is true for any function $p(\rho)$ satisfying (6.2).

By analogy with the gas dynamics equations or the unsteady transonic small disturbance equation, we can think of U_R and U_F as "weak" and "strong" regular reflection, respectively.

Appendix B: Structural Conditions for the Fixed Boundary Problem

We state the structural conditions of Section 4.3 [15] in the notation of this paper.

First, we define

$$\tilde{f}(\xi, \eta) := \begin{cases} f(\xi, \eta), & (\xi, \eta) \in \sigma, \\ \rho_s, & (\xi, \eta) = \Xi_s. \end{cases} \tag{6.10}$$

By (2.17) we have

$$\rho_0 + \epsilon_* \le \tilde{f} \le \rho_s,$$

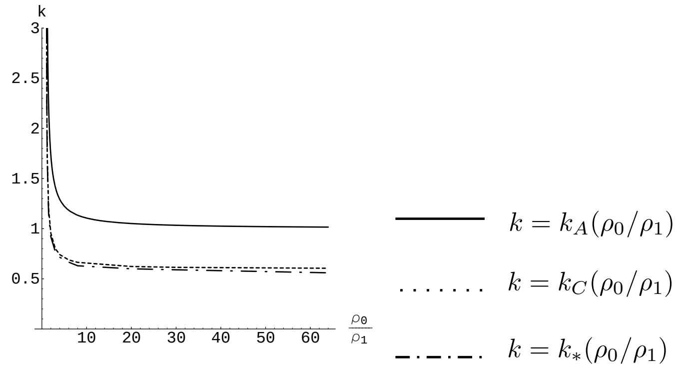

FIGURE 8. The curves k_A, k_C and k_* for the γ-law pressure, $\gamma = 2$.

which implies bounds on $\tilde{f}$ independent of $r \in \mathcal{K}$ and of ρ. (This is the first condition in (4.8) [15].) Note that, in the context of the NLWS, the second condition in (4.8) [15] on the function $\tilde{f}$ is replaced by

$$c^2(\tilde{f}(r)) > r^2 \quad \text{on} \quad \sigma = \{(r, \theta^*) : 0 \le r \le r(\theta^*)\},$$

ensuring that the solution is subsonic (as we have stated in (2.17)).

Next, we introduce the boundary operator $\tilde{N}$ on $\Sigma \cup \Sigma_0$ as

$$\tilde{N}(\rho) := \tilde{\chi} \cdot \nabla \rho, \tag{6.11}$$

where the vector $\tilde{\chi}$ is defined by

$$\tilde{\chi} := \begin{cases} \chi, & \text{on} \quad \Sigma, \\ (1, 0), & \text{on} \quad \Sigma_0, \end{cases} \tag{6.12}$$

with χ given by (3.12).

Then the fixed boundary value problem (3.18) can be written as

$$\begin{array}{lll} \tilde{Q}(\rho) = 0 & \text{in} & \Omega, \\ \tilde{N}(\rho) = 0 & \text{on} & \tilde{\Sigma} := \Sigma \cup \Sigma_0, \\ \rho = \tilde{f} & \text{on} & \partial\Omega \setminus \tilde{\Sigma} = \sigma \cup \Xi_s, \end{array} \tag{6.13}$$

where $\tilde{Q}$ is the operator given in (3.3), $\tilde{N}$ is given by (6.11) and $\tilde{f}$ is given by (6.10). Moreover, we write the operator $\tilde{Q}$ as

$$\tilde{Q}(\rho) = \sum_{i,j} a_{ij}(\rho, \xi, \eta) D^{ij}\rho + \sum_{i} b_i(\rho, \xi, \eta) D^i \rho + \sum_{i,j} c_{ij}(\rho, \xi, \eta) D^i \rho D^j \rho. \tag{6.14}$$

The structural conditions imposed on the problem (6.13) in [15] are as follows.

- The coefficients a_{ij}, b_i and c_{ij} are in C^1, and for a fixed curve $r \in \mathcal{K}$ we have $\tilde{\chi}_i \in H_{\alpha_\Sigma}$.
- The operator $\tilde{Q}$ is strictly elliptic, meaning

$$\lambda \ge C_1 > 0, \qquad \text{for all } \rho \text{ and } r \in \mathcal{K}, \tag{6.15}$$

where λ denotes the smallest eigenvalue of the operator $\tilde{Q}$. We also assume a bound on the ellipticity ratio of the form

$$\frac{\Lambda}{\lambda} \leq C_2(|\rho|_0), \qquad \text{for all } \rho \text{ and } r \in \mathcal{K}, \tag{6.16}$$

where $C_2(|\rho|_0)$ is a continuous function on $\mathbb{R}^+$. Here, Λ denotes the maximum eigenvalue of $\tilde{Q}$.

- The operator $\tilde{N}$ is strictly oblique, i.e.,

$$\tilde{\chi} \cdot \nu \geq C_3 > 0, \qquad \text{for all } \rho \text{ and } r \in \mathcal{K}, \tag{6.17}$$

where ν stands for the unit inward normal to the boundary $\tilde{\Sigma}$. Also,

$$|\tilde{\chi}| \leq C_4(|\rho|_0), \qquad \text{for all } \rho \text{ and } r \in \mathcal{K}, \tag{6.18}$$

holds, where $C_4(|\rho|_0)$ is a continuous function on $\mathbb{R}^+$.

- For any solution ρ to the equation $\tilde{Q}(\rho) = 0$ in Ω we have

$$0 \leq \sum_{i,j} c_{ij}(\rho, \xi, \eta) D^i \rho D^j \rho, \tag{6.19}$$

and there exist $\mu_0, \Phi \in \mathbb{R}$, independent of ρ, such that

$$|\sum_{i,j} a_{ij}(\rho, \xi, \eta) D^{ij} \rho| \leq \lambda \left(\mu_0 \sum_i |D^i \rho|^2 + \Phi \right). \tag{6.20}$$

It is noted in Remark 4.7 of [15] that, under the above conditions, a uniform bound on the supremum norm $|\rho|_0$, where ρ is any solution to the equation $\tilde{Q} = 0$ in Ω, implies the following:

- the norms $|a_{ij}|_0$, $|b_i|_0$, $|c_{ij}|_0$ and $|\chi_i|_0$ are uniformly bounded in ρ and $r \in \mathcal{K}$, and a uniform bound on the α-Holder seminorm $[\rho]_\alpha$ implies that $[a_{ij}]_\alpha$, $[b_i]_\alpha$, $[c_{ij}]_\alpha$ and $[\chi_i]_\alpha$ are uniformly bounded in ρ and $r \in \mathcal{K}$ (here, $\alpha \in (0,1)$ is arbitrary),
- the operator $\tilde{Q}$ is uniformly elliptic,
- the boundary operator $\tilde{N}$ is uniformly oblique, and
- since the matrix $[a_{ij}(\rho, \xi, \eta)]$ is uniformly positive definite and the coefficients $c_{ij}(\rho, \xi, \eta)$ are uniformly bounded, there exists $K > 0$, independent of ρ and $r \in \mathcal{K}$, such that

$$\sum_{i,j} c_{ij}(\rho, \xi, \eta) D^i \rho D^j \rho \leq K \sum_{i,j} a_{ij}(\rho, \xi, \eta) D^i \rho D^j \rho. \tag{6.21}$$

(This constant K plays role in the construction of a subsolution to the nonlinear fixed boundary problem (6.13) which is used to show that a solution to (6.13) exists. For more details, see Lemma 4.14 in [15]).

In [15] the additional condition $\Sigma \subset H_{2+\alpha}$, where $\alpha \in (0,1)$, was imposed. However in steps 3 and 4 of the proof of Theorem 4.11 we showed how to eliminate this condition. This, we require only the weaker hypothesis $\Sigma \in H_{1+\alpha_\Sigma}$ leading to $\tilde{\chi} \in H_{\alpha_\Sigma}$.

Appendix C: Definitions of Weighted Hölder Spaces

For a set $S \subseteq \mathbb{R}^2$ and a function $u : S \to \mathbb{R}$, we recall the definitions of the following seminorms and norms (for more details see [14]):

$$\begin{array}{ll} |u|_{0;S} := \sup_{X\in S} |u(X)| & \text{supremum norm,} \\ [u]_{\alpha;S} := \sup_{X\neq Y} \frac{|u(X)-u(Y)|}{|X-Y|^\alpha} & \alpha\text{-Holder seminorm,} \\ |u|_{\alpha;S} := |u|_{0;S} + [u]_{\alpha;S} & \alpha\text{-Holder norm,} \\ |u|_{k+\alpha;S} := \sum_{i=0}^k |D^i u|_{0;S} + [D^k u]_{\alpha;S} & (k+\alpha)\text{-Holder norm.} \end{array}$$

Here, $\alpha \in (0,1)$, k is a nonnegative integer and $D^i u$ denotes the collection of the $i-$th order derivatives of u.

In the definition of the set $\mathcal{K}$ and in Theorem 2.3, $H_{1+\alpha}$ denotes the space of all curves $r(\theta)$, $\theta \in (\theta^*, \pi/2)$, such that

$$|r|_{1+\alpha;(\theta^*,\pi/2)} < \infty,$$

and $H_{1+\alpha}^{(-\gamma)}$ denotes the space of functions $u(\xi,\eta)$, $(\xi,\eta)\in\Omega$, such that

$$|u|^{(-\gamma)}_{1+\alpha;\Omega\setminus\mathbf{V}} := \sup_{\delta>0} \delta^{1+\alpha-\gamma} |u|_{1+\alpha;\Omega_{\delta;\mathbf{V}}} < \infty,$$

with $\Omega_{\delta;\mathbf{V}} := \{X \in \Omega : \mathrm{dist}(X,\mathbf{V}) > \delta\}$ and $\mathbf{V} := \{O, V, \Xi_s\}$.

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