Quiz 10. HW due April 15 on Linear Algebra 5.5 - 5.8.

Problems that are graded:

5.6	# 18	3 pts
5.7	# 6	3 pts
5.8	# 13	4 pts
		10 pts

Below are the solutions to all problems that were assigned:

5.5

2) $A = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ 3×1 $\quad B = \begin{pmatrix} 0 & -2 & 3 \\ 3 & -2 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ 3×3 $\quad C = \begin{pmatrix} -2 & 4 & -3 \end{pmatrix}$ 1×3

$D = \begin{pmatrix} -6 & 5 \\ 3 & 0 \\ -4 & -4 \end{pmatrix}$ 3×2

(a) AB $\quad (3 \times 1)(3 \times 3)$ not possible

(b) BA $\quad (3 \times 3)(3 \times 1) \rightarrow (3 \times 1)$

$$BA = \begin{pmatrix} 0 & -2 & 3 \\ 3 & -2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4+12 \\ -3-4 \\ -2+6+4 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ 8 \end{pmatrix}$$

(c) CB $\quad (1 \times 3)(3 \times 3) \rightarrow (1 \times 3)$

$$CB = \begin{pmatrix} -2 & 4 & -3 \end{pmatrix} \begin{pmatrix} 0 & -2 & 3 \\ 3 & -2 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 12-6 & 4-8-9 & -6-3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -13 & -9 \end{pmatrix}$$

(d) $CA = \begin{pmatrix} -2 & 4 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = 2 + 8 - 12 = -2$

(e) DA not possible

(f) DB not possible

(g) $AC = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} -2 & 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -4 & 3 \\ -4 & 8 & -6 \\ -8 & 16 & -12 \end{pmatrix}$

(h) $B^2 = \begin{pmatrix} 0 & -2 & 3 \\ 3 & -2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 3 \\ 3 & -2 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -6+6 & 4+9 & 3 \\ -6 & -6+4 & 9 \\ 9+2 & -4-6+3 & 6+1 \end{pmatrix} = \begin{pmatrix} 0 & 13 & 3 \\ -6 & -2 & 9 \\ 11 & -7 & 7 \end{pmatrix}$$

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18) $-x+y=5$
$-x+z=-2$
$6x-2y-3z=1$

$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix}$

$$(A|I) = \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 4 & -3 & 6 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right) \quad A^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} b = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10-6+1 \\ 15-6+1 \\ 10-8+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad x = 5$$
$$y = 10$$
$$z = 3$$

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22)

across the 3rd row

$$\begin{vmatrix} 2 & -3 & 0 \\ 5 & -2 & 0 \\ 2 & 0 & -1 \end{vmatrix} = 2 \begin{vmatrix} -3 & 0 \\ -2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix}$$

$$= 2 \cdot 0 - (-4+15) = -11$$

or down the 3rd column

$$= -1 \cdot \begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix} = -(-4+15) = -11$$

5.6

34)

$$3x + z = -2$$
$$x + 2y - z = 0$$
$$x - 4y + 3z = 1$$

$z = ?$

Cramer's rule: $A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{pmatrix}$

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{vmatrix} = 3 \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -4 \end{vmatrix} = 3(6-4) + (-4-2)$$
$$= 6 - 6 = 0$$

Since $\det A = 0$, Cramer's rule does not apply.

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39) $\begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 30 = 2 - \lambda - 2\lambda + \lambda^2 - 30$

$= \lambda^2 - 3\lambda - 28 = (\lambda + 4)(\lambda - 7)$

The system $(1-\lambda)x + 6y = 0$

$5x + (2-\lambda)y = 0$

has non-trivial solutions $\Longleftrightarrow$ the matrix $A = \begin{pmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{pmatrix}$ is not invertible

$\Longleftrightarrow$ $\det A = 0$ $\Longleftrightarrow$ $\lambda = -4$ or 7

If $\lambda = -4$: $\left(\begin{array}{cc|c} 5 & 6 & 0 \\ 5 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ $x = -6/5\, y$ $\Rightarrow$ solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} -6/5 \\ 1 \end{pmatrix}$

If $\lambda = 7$: $\left(\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ $x = y$ solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

where a is a free parameter

5.7) (6) $\{\underset{v_1}{(1,2,-3)}, \underset{v_2}{(1,-3,2)}, \underset{v_3}{(2,-1,5)}\}$ in $\mathbb{R}^3$ lin. indep or dep?

There are 3-ways to do the problem:

(A) $\left(\begin{array}{ccc|c} v_1 & v_2 & v_3 & \begin{matrix}0\\0\\0\end{matrix}\end{array}\right)$ row operation to check whether there are non-trivial solutions

(B) $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ row operation to see if one can get a row of zeros.

(C) $\det\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ or $\det\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \overset{?}{=} 0$

We choose (c):

$$\det\begin{pmatrix} 1 & 1 & 2 \\ 2 & -3 & 1 \\ -3 & 2 & 5 \end{pmatrix} = -15+3+8-18+2-10$$

$$= -12-10-8 \neq 0$$

$$\Rightarrow \{v_1, v_2, v_3\} \text{ lin. indep.}$$

(14) $v_1 = (1, b, 2b)$ $v_2 = (2,1,4)$ in $\mathbb{R}^3$ lin. indep?

They are lin. dependent if

$$\frac{2}{1} = \frac{1}{b} = \frac{4}{2b} \qquad \frac{2}{1} = \frac{1}{b} \Rightarrow b = 1/2$$

$$\frac{2}{1} = \frac{4}{2b} \Rightarrow b = 1$$

$\Rightarrow$ they cannot be lin. dependent for any value of b

So v_1, v_2 are lin. indep. for <u>all</u> values of b

22) $f_1 = x$

$f_2 = e^x$

$f_3 = xe^x$

$$W(f_1, f_2, f_3) = \begin{vmatrix} x & e^x & xe^x \\ 1 & e^x & e^x + xe^x \\ 0 & e^x & e^x + e^x + xe^x \end{vmatrix}$$

$$= \begin{vmatrix} x & e^x & xe^x \\ 1 & e^x & (x+1)e^x \\ 0 & e^x & (x+2)e^x \end{vmatrix}$$

$$= x \begin{vmatrix} e^x & (x+1)e^x \\ e^x & (x+2)e^x \end{vmatrix} - \begin{vmatrix} e^x & xe^x \\ e^x & (x+2)e^x \end{vmatrix}$$

$$= x\left(\cancel{x}+2-\cancel{x}-1\right)e^{2x} - \left(\cancel{x}+2-\cancel{x}\right)e^{2x}$$

$$= xe^{2x} - 2e^{2x} = (x-2)e^{2x} \neq 0$$

$\{f_1, f_2, f_3\}$ lin. independent

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8) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\lambda_1 = 1+i$, e-vector v_1

$\lambda_2 = 1-i$, e-vector $v_2 = \overline{v_1}$

Set $\lambda_1 = 1+i$ and $v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1-(1+i) & -1 & 0 \\ 1 & 1-(1+i) & 0 \end{array}\right) = \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - i x_2 = 0$

$\begin{cases} x_1 = i x_2 \\ x_2 = x_2 \end{cases}$

$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda_1 = 1+i$

Then $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ $\lambda_2 = 1-i$

13) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{pmatrix}$ Hint: $\lambda = 1$ e-value

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -2 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 2-\lambda \\ -2 & -2 \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)(-1-\lambda)+2\right] + (1+\lambda) - 2 - 2 + 2(2-\lambda)$$

$$= (2-\lambda)^2(-1-\lambda) + 4 - 2\lambda + 1 + \lambda - \cancel{4} + \cancel{4} - 2\lambda$$

$$= -(4+\lambda^2-4\lambda)(1+\lambda) + 5 - 3\lambda$$

$$= -4 - \cancel{4\lambda} - \lambda^2 - \lambda^3 + \cancel{4\lambda} + 4\lambda^2 + 5 - 3\lambda$$

$$= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda-1)^3 \qquad \lambda = 1 \text{ mult. } 3$$

<u>E-vectors:</u> $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 = -x_2 - x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ e-vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

corresponding to $\lambda = 1$