QUIZ 12

Math 3321

Name: Solutions

Due Date: April 22, 2008

April 17, 2008

Question 1. *(5pts)*

(i) Find the general solution of the system

$$\mathbf{x} = \begin{pmatrix} 3 & -4 & 4 \\ 4 & -5 & 4 \\ 4 & -4 & 3 \end{pmatrix} \mathbf{x}.$$

(ii) Find the solution that satisfies the above system of equations and the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

E-values

$$\begin{vmatrix} 3-\lambda & -4 & 4 \\ 4 & -5-\lambda & 4 \\ 4 & -4 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(-5-\lambda) - 64 - 64 + (5+\lambda)16 + 16(3-\lambda) + 16(3-\lambda) =$$

$$= (3-\lambda)^2(-5-\lambda) - 128 + 16(6-2\lambda+5+\lambda)$$

$$= (3-\lambda)^2(-5-\lambda) - 128 + 16(11-\lambda)$$

$$= (9+\lambda^2-6\lambda)(-5-\lambda) - 128 + 176 - 16\lambda$$

$$= -45 - 9\lambda - 5\lambda^2 - \lambda^3 + 30\lambda + 6\lambda^2 + 48 - 16\lambda$$

$$= -\lambda^3 + \lambda^2 + 5\lambda + 3 = -(\lambda+1)^2(\lambda-3)$$

$\lambda_1 = \lambda_2 = -1 \quad \lambda_3 = 3$

$\lambda_1 = \lambda_2 = -1$

$$\left(\begin{array}{ccc|c} 4 & -4 & 4 & 0 \\ 4 & -4 & 4 & 0 \\ 4 & -4 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} y_1 = y_2 - y_3 \\ y_2 = a \\ y_3 = b \end{array}$$

e-vectors $\quad z_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad z_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = 3$

$$\left(\begin{array}{ccc|c} 0 & -4 & 4 & 0 \\ 4 & -8 & 4 & 0 \\ 4 & -4 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{e-vector} \qquad z_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Gen. solution: $\quad x = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\quad x(0) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{aligned} c_1 - c_2 + c_3 &= 2 \\ c_2 &= -1 \\ c_3 &= 0 \end{aligned} \qquad \begin{aligned} c_1 &= c_2 + 2 = -1 + 2 = 1 \\ c_2 &= -1 \\ c_3 &= 0 \end{aligned}$$

Solution:

$$x = e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = e^{-t} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Question 2. *(5pts) Find the general solution of the system*

$$\mathbf{x}' = \begin{pmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{pmatrix} \mathbf{x}.$$

$$A = \begin{pmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 4 & 0 \\ -1 & -\lambda & 0 \\ 1 & 4 & -1-\lambda \end{vmatrix} = \lambda^2(-1-\lambda) + 4(-1-\lambda) = (\lambda^2+4)(-1-\lambda)$$

$$\lambda = \pm 2i \qquad \lambda = -1$$

$\underline{\lambda_1 = -1:}$ $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$y_1 + 4y_2 = 0 \qquad y_1 = 0$

$y_2 = 0 \qquad y_2 = 0$

$y_3 = a$

e-vector $z_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 2i:}$ $\left(\begin{array}{ccc|c} -2i & 4 & 0 & 0 \\ -1 & -2i & 0 & 0 \\ 1 & 4 & -1-2i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4-2i & -1-2i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 4-2i & -1-2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 1 & \frac{-1-2i}{4-2i} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad y_1 = -2i\left(\frac{i}{2}a\right)$, $y_2 = +\frac{i}{2}a$, $y_3 = a$

$y_1 = a$

$y_2 = \frac{i}{2}a$

$y_3 = a$

e-vector $z_2 = \begin{pmatrix} 1 \\ i/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}$

<u>General solution</u>

$$x = c_1 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \left[\cos 2t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}\right] + c_3 \left[\cos 2t \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right]$$