Quiz 4: HW due 2/7/08

2.1: (22) $xy' + y = xe^x \quad y(-1) = e^{-1}$

$$y' + \frac{1}{x}y = e^x$$

$$\int \frac{1}{x}dx = \ln x \qquad e^{\ln x} = x$$

$$(xy)' = xe^x$$

$$xy = \int xe^x\,dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$\boxed{y = e^x - \frac{1}{x}e^x + \frac{C}{x}}$$

$y(-1) = e^{-1} \qquad e^{-1} = e^{-1} + e^{-1} - C \qquad C = e^{-1}$

$$\boxed{y = e^x - \frac{1}{x}e^x + \frac{e^{-1}}{x}}$$

2.2 (31) $\dfrac{dy}{dx} = \dfrac{x^2 e^{y/x} + y^2}{xy} = f(x,y) \qquad (*)$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 e^{\lambda y/\lambda x} + \lambda^2 y^2}{\lambda x \lambda y} = \frac{\lambda^2(e^{y/x} + y^2)}{\lambda^2 xy} = \frac{e^{y/x} + y^2}{xy} = f(x,y)$$

$\Rightarrow$ homogeneous.

Set $y = u \cdot x$: Substitute in $(*)$

$$u'x + u = \frac{x^2 e^u + u^2 x^2}{x^2 u} = \frac{e^u + u^2}{u}$$

$$u'x + u = \frac{e^u + u^2}{u}$$

$$u'x = \frac{e^u + u^2}{u} - u = \frac{e^u + u^2 - u^2}{u}$$

$$u'x = \frac{e^u}{u}$$

$$\frac{u\,u'}{e^u} = \frac{1}{x}$$

$$\int u e^{-u}\,du = \int \frac{1}{x}\,dx$$

$$-u e^{-u} + \int e^{-u}\,du = \ln|x| + C$$

$$-u e^{-u} - e^{-u} = \ln|x| + C$$

$$\boxed{(u+1)\,e^{-u} = C - \ln|x|} \qquad u = \frac{y}{x}$$

$$\boxed{\left(\frac{y}{x} + 1\right) e^{-y/x} = C - \ln|x|}$$

$$\frac{y + x}{x\,e^{y/x}} = C - \ln|x|$$

or

$$\boxed{\frac{y+1}{e^{y/x}} = Cx - x\ln|x|}$$

2.3.1 (4)

(1) $y^2 = 2(C-x)$

Diff. eq of (1): $2yy' = 2\cdot(-1)$

$$yy' = -1$$

$$y' = -\frac{1}{y}$$

New diff. eq.: $y' = y$

$$\frac{1}{y}y' = 1$$

$$\ln|y| = x + C$$

$$\boxed{y = C'e^x}$$ orthogonal trajectories

2.3.6 (1) 5000 population

$P(t) = \#$ aware of the rumor

$P(0) = 100$ $\quad P'(t) = kP(5000-P)$

$P(3) = 500$

(a) $P(8) = ?$

$$P' = kP(5000-P)$$

$$\int \frac{dP}{P(5000-P)} = kt + C$$

$$\frac{1}{5000}\int\left(\frac{1}{P} + \frac{1}{5000-P}\right) dP = kt + C$$

$$\ln(P) - \ln(5000-P) = 5000kt + C$$

$$\ln\left(\frac{P}{5000-P}\right) = 5000kt + C$$

$$\frac{P}{5000-P} = A e^{5000kt}$$

$$P(0)=100 \Rightarrow \frac{1\cancel{00}}{4900} = A \qquad \boxed{A = 1/49}$$

$$P(3) = 500 \qquad \frac{5\cancel{00}}{45\cancel{00}} = \frac{1}{49} e^{15000k}$$

$$\ln\frac{49}{9} = 15000k \qquad \boxed{k = \frac{1}{15000}\ln\left(\frac{49}{9}\right)}$$

$$(**) \quad \frac{P}{5000-P} = \frac{1}{49} e^{\frac{t}{3}\ln\left(\frac{49}{9}\right)}$$

$$\frac{P}{5000-P} = \frac{1}{49}\left(\frac{49}{9}\right)^{t/3} \Rightarrow P = \frac{1}{49}\left(\frac{49}{9}\right)^{t/3}(5000-P)$$

$$P\left(1+\frac{1}{49}\left(\frac{49}{9}\right)^{t/3}\right) = \frac{5000}{49}\left(\frac{49}{9}\right)^{t/3}$$

$$P = \frac{5000\left(49/9\right)^{t/3}}{49\left(1+\frac{1}{49}\left(\frac{49}{9}\right)^{t/3}\right)}$$

$$P(8) = \frac{5000\left(49/9\right)^{8/3}}{49\left(1+\frac{1}{49}\left(\frac{49}{9}\right)^{8/3}\right)} \approx 3259 \text{ people}$$

(b) $t = ?$ $\quad P = 2500$

Substitute in (**)

$$\frac{2500}{5000 - 2500} = \frac{1}{49} e^{\frac{t}{3} \ln\left(\frac{49}{9}\right)}$$

$$1 = \frac{1}{49} e^{\frac{t}{3} \ln\left(\frac{49}{9}\right)}$$

$$\ln 49 = \frac{t}{3} \ln\left(\frac{49}{9}\right) \qquad t = \frac{3 \ln 49}{\ln(49/9)} \approx 6.89 \text{ days}$$