Sharp Decay Estimates for Hyperbolic Balance Laws

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Abstract

We consider strictly hyperbolic and genuinely nonlinear systems of hyperbolic balance laws in one space dimension. Sharp decay estimates are derived for the positive waves in an entropy weak solution. The result is obtained by introducing a partial ordering within the family of positive Radon measures, using symmetric rearrangements and a comparison with a solution of Burgers's equation with impulsive sources as well as lower semicontinuity properties of continuous Glimm-type functionals.

Key words: Hyperbolic balance laws, total variation, interaction potential, Glimm functionals, front tracking algorithm, spreading of rarefaction waves, genuine nonlinearity, BV estimates, dissipative.
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1 Introduction

We study the spreading of rarefaction waves for systems of balance laws in one-space dimension

$$\partial_t u + \partial_x f(u) + g(u) = 0 \ . \tag{1.1}$$

Here $x \in \mathbb{R}$, $u(t,x) \in \mathbb{R}^n$, f and g are smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. We assume that system (1.1) is strictly hyperbolic, that is, the Jacobian matrix

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$A(u) = Df(u)$ has n real and distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u), \tag{1.2}$$

known as characteristic speeds. Let $< l_i(u), r_i(u) >$ denote the corresponding left and right linearly independent eigenvectors associated with these characteristic speeds normalized so that

$$l_i(u) r_j(u) = \begin{cases} 1, \text{ if } & i = j, \\ 0, \text{ if } & i \neq j. \end{cases} \tag{1.3}$$

Here, and in what follows, we assume that all characteristic fields are genuinely nonlinear, namely for each $i = 1, \ldots, n$, the product of the gradient of the characteristic speed with the corresponding right eigenvector is nonzero. Without loss of generality, this condition can be expressed as

$$D\lambda_i(u) r_i(u) \equiv 1. \tag{1.4}$$

The assumption of genuine nonlinearity plays a crucial role in many aspects of the forthcoming analysis.

The aim of this paper is to establish a sharp estimate that allows to compare the decay of the measure of positive waves for each characteristic family of (1.1) with the distributional derivative of a scalar balance law with a forcing term.

Systems of conservation laws, i.e. $g \equiv 0$, with Cauchy data

$$u(0, x) = u_0(x) \tag{1.5}$$

have been studied extensively. Such equations arise in elasticity, continuum mechanics, magnetohydrodynamics and other applied areas. Global existence has been established when the initial data has small total variation. More precisely, four different techniques have been developed for constructing global weak solutions, namely the random choice method of Glimm, the front tracking method, the vanishing viscosity method and the functional analytic method of compensated compactness. Expositions of the current state of the theory together with relevant bibliography may be found in the books [7,16,32,33].

For systems of balance laws with general source term g, we expect blow-up of solutions in finite time even when the initial data is of small variation. Indeed, because of the presence of the production term $g(u)$, small oscillations in the solution may amplify in time, hence in general one does not have long term stability in BV. Local in time existence of BV solutions was first established by Dafermos and Hsiao [17], by the random choice method of Glimm [18]. On the other hand, global existence is expected if g satisfies some special mechanism

that induces dissipation. Dafermos and Hsiao in [17] also investigated this situation and constructed global BV solutions under a suitable dissipativeness assumption on g using the same approximating procedure. Under the same dissipativeness assumption, existence as well as stability results have been established via the front tracking method by Amadori and Guerra [2] for hypebolic balance laws whose characteristic fields are either genuine nonlinear or linearly degenerate. These wellposedness results have also been established for hyperbolic balance laws with general flux, even in the case of non-conservative systems, via the method of vanishing viscosity by Christoforou [13].

The dissipativeness hypothesis on the source g is the following: Consider a constant equilibrium solution u^* to (1.1), i.e. $g(u^*) = 0$. If we linearize the hyperbolic system (1.1) about u^* and then decompose the solution u along the right eigenvectors of $A(u^*)$, the resulting linear system is

$$v_{i,t} + \lambda_i(u^*)v_{i,x} + \sum_{j=1}^{n} B_{ij}(u^*)v_j = 0 \ , \tag{1.6}$$

where B_{ij} are the entries of the $n \times n$ matrix

$$B(u) = [r_1(u), ..., r_n(u)]^{-1} Dg(u)[r_1(u), ..., r_n(u)]. \tag{1.7}$$

The natural condition that renders the above linear system stable in L^1 is that the matrix $B(u^*)$ is *strictly column diagonally dominant*, i.e. there is a positive constant $\tilde{\beta}$ such that

$$B_{ii}(u^*) - \sum_{j \neq i} |B_{ji}(u^*)| > \tilde{\beta} > 0 \qquad \text{for all } i = 1, ..., n, \tag{1.8}$$

which is equivalent to

$$|I - \tau B(u^*)| < 1 - \tilde{\beta}\tau \ , \tag{1.9}$$

for every $\tau \geq 0$ sufficiently small and $|\cdot|$ stands for the l^1 norm of a matrix. It is obvious that this hypothesis depends on the choice of the right eigenvectors $r_i(u)$. Therefore, for practical use, one can find an equivalent condition in [1] which is independent of the choice of the eigenvectors. We remark that such condition holds in the case of frictional damping, but for $\tilde{\beta} = 0$.

In addition to global existence and stability results [17,2,13] to (1.1) under the diagonally dominance condition (1.8), the uniqueness of BV solutions within the class of *viscosity solutions* is presented in Christoforou [14]. This result implies that the entropy weak solutions constructed using the three different schemes (the random choice, the front tracking and the vanishing viscosity method) coincide.

It should be added that the results in [17] are obtained when the flux f and the source g can depend explicitly upon (t, x) under additional assumptions

on Df and g. We also refer to Liu [27] for global existence results to (1.1) for a different class of source terms.

A sharp decay estimate on the spreading of rarefaction waves was established by Bressan and Yang for hyperbolic systems of conservation laws without source term $g(u) = 0$ in [11]. This sharp estimate has proven to be extremely important in providing information on the rate of convergence for vanishing viscosity approximations for hyperbolic systems of conservation laws, cf. [12]. See also [4] and the references therein. The aim of this paper is to show that sharp decay estimates can be established even in the case of hyperbolic balance laws (1.1), to analyze the effect of the source g and to provide a comparison with a solution to the Burgers's equation with impulsive sources.

Related results for *approximate* solutions to hyperbolic systems of conservation laws constructed by the random choice method of Glimm were given by Glimm and Lax [19], Lax [22,23] (genuinely nonlinear systems) and Liu [28] (piecewise genuinely nonlinear systems). An estimate on the spreading of rarefaction waves for *general BV solutions* of genuinely nonlinear systems of hyperbolic conservation laws was established by Bressan and Colombo [8], and Bressan [7]. Glimm-type estimates for general BV solutions were also presented by Dafermos [16] and Trivisa [34] for hyperbolic systems of two conservation laws using the method of *generalized characteristsics*. We also refer to Dafermos [15] and Jenssen and Sinestrari [21] for scalar conservation laws with one inflection point, to the article by Ancona and Marson [3] for the study of systems of two conservation laws with a single inflection point and to LeFloch and Trivisa [25] for hyperbolic systems of conservation laws with a more general flux-function. The spreading estimate controls the decay of solutions for large times. Further results on the large-time decay of solutions, especially for systems of two conservation laws, can be found in Dafermos [16].

In the next subsection, we introduce a notion of partial ordering within the family of positive Radon measures, define the positive i-measures and the Glimm functionals of a function of bounded variation and in Subsection 1.2, we present the main results.

1.1 Preliminaries

Let $u(t, \cdot)$ be a function of bounded variation, as such its distributional derivative $D_x u$ is a Radon measure. Here and in what follows, $u(t, \cdot)$ will denote a function of small total variation. Following [7], we define the wave measures μ^i of i-waves in u: Consider the distributional derivative $D_x u$ and define

$$\mu^i = l_i(u) \cdot D_x u$$

on the sets of continuity of u and

$$\mu^i(\{x\}) = \sigma_i$$

at the points of jump of u. Here, σ_i denotes the strength of the i-wave in the Riemann solution with data the left $u_l = u(x-)$ and right $u_r = u(x+)$ states. If we denote by $u_l = u_0$, $u_1, \ldots, u_n = u_r$ the states in the Riemann solution, and take into consideration the fact that all characteristic fields are genuine nonlinear, then we can define the strength of the i-wave to be

$$\sigma_i = \lambda_i(u_i) - \lambda_i(u_{i-1}) \ .$$

Taking into account that

$$\sigma_i = l_i(u_r) \cdot (u_r - u_l) + O(1) \cdot |u_r - u_l|^2,$$

one can identify a vector $l_i(x)$ such that

$$\begin{cases} |l_i(x) - l_i(u_r)| = O(1) \cdot |u_r - u_l|, \\ \sigma_i = l_i(x) \cdot (u_r - u_l). \end{cases} \tag{1.10}$$

Therefore, we can now define the measure μ^i as

$$\mu^i = l_i \cdot D_x u, \tag{1.11}$$

where $l_i(x) = l_i(u(x))$, at points of continuity for u, while at jump points $l_i(x)$ is defined through relations (1.10).

Throughout this paper, we use $O(1)$ to denote a large positive constant.

We denote by μ^{i+} and μ^{i-} the positive and negative parts of μ^i and by $|A|$ the Lebesgue measure of a set A.

Definition 1.1 *Let μ, μ' be two positive Radon measures. We say that*

$$\mu \preceq \mu' \quad \textit{if and only if} \quad \sup_{|A| \le s} \mu(A) \le \sup_{|B| \le s} \mu'(B) \ , \tag{1.12}$$

for every $s > 0$.

We remark that the usual relation

$$\mu \le \mu' \quad \text{if and only if} \quad \mu(A) \le \mu'(A) \tag{1.13}$$

for every $A \subset \mathbb{R}$, is stronger, in the sense that it impies $\mu \preceq \mu'$, but the reverse does not hold.

In association with the measures μ^i, we define the Glimm functionals

$$V(u) = \sum_i |\mu^i|(\mathbb{R}), \tag{1.14}$$

$$Q(u) = \sum_{i<j} (|\mu^j| \otimes |\mu^i|)\{(x,y); x < y\} + \sum_i (\mu^{i-} \otimes |\mu^i|)\{(x,y); x \neq y\}. \tag{1.15}$$

Let us denote by $u = u(x,t)$ an entropy solution of system (1.1). Under the dissipativeness condition (1.8), if the total variation of u is small and the constant c is large enough, the functionals

$$Q(t) = Q(u(t)), \quad \Upsilon(t) = V(u(t)) + cQ(u(t)) \tag{1.16}$$

are nonincreasing in time and more precisely, they are decaying exponentially fast. The decrease in Q controls the amount of interaction, while the decrease in Υ controls both the interaction and the cancellation in the solution. A sharp estimate on the measure μ_t^{i+} of positive i-waves will be obtained by a comparison with a solution of Burgers's equation with source terms.

1.2 Main results

In this section, we present the main results of the article. We start our presentation by stating two theorems. The first refers to hyperbolic balance laws (1.1) under the assumption that the dissipativeness condition (1.8) is satisfied for the source term $g = g(u)$, while the latter refers to the case of a general source term $g = g(u)$. We remark, that in the more general case there is a price to be paid, in the sense that the result is *local*. For hyperbolic balance laws with a general source we have, in general, *blow-up* of weak solutions in *finite* time. Therefore, the decay estimate in that case, holds within the time interval of existence of solutions.

Theorem 1.2 *Consider system of balance laws* (1.1) *with initial data* (1.5) *and assume that the system is strictly hyperbolic, each characteristic field is genuinely nonlinear and the dissipativeness hypothesis* (1.8) *holds.*

Let $w = w(t,x)$ be the solution of the Cauchy problem for the scalar Burgers's equation with impulsive source term

$$\partial_t w + \partial_x \left(\frac{w^2}{2}\right) + \beta w = -\kappa sgn(x) \cdot \frac{d}{dt} Q(u(t)), \tag{1.17}$$

$$w(0,x) = sgn(x) \cdot \sup_{|A|<2|x|} \frac{\mu_0^{i+}(A)}{2} \tag{1.18}$$

with β, κ denoting some positive constants.

Then, for every solution $u = u(t,x)$ of (1.1), (1.5) *with small total variation, and for every $t \geq 0$, and $i = 1, \ldots, n$,*

$$\mu_t^{i+} \preceq D_x w(t). \tag{1.19}$$

The following theorem states the result in the general case of a local in time BV solution to system (1.1), i.e. when hypothesis (1.8) does not hold necessarily.

Theorem 1.3 (Local in Time) *Consider system of balance laws* (1.1) *with initial data* (1.5) *and assume that the system is strictly hyperbolic, each characteristic field is genuinely nonlinear. Denote by $T > 0$ the maximal time of existence of a solution to* (1.1), (1.5).

Let $w = w(t,x)$ be the solution of the Cauchy problem for the scalar Burgers's equation with impulsive source term

$$\partial_t w + \partial_x \left(\frac{w^2}{2} \right) = \gamma w - \kappa sgn(x) \cdot \frac{d}{dt} Q(u(t)), \tag{1.20}$$

$$w(0,x) = sgn(x) \cdot \sup_{|A|<2|x|} \frac{\mu_0^{i+}(A)}{2} \tag{1.21}$$

with γ, κ denoting some positive constants.

Then, for every solution $u = u(t,x)$ of (1.1), (1.5) *with small total variation, and for every $t \in [0,T]$ and $i = 1, \ldots, n$,*

$$\mu_t^{i+} \preceq D_x w(t). \tag{1.22}$$

As the Burgers's equation represents the prototype example for hyperbolic conservation laws, the Burgers's equation with impulsive source terms represents the prototype example for hyperbolic balance laws. In some interesting, yet special, cases the characteristic speeds of systems of conservation laws are due to (1.4) solutions of Burgers's type equations.

The outline of this article is as follows: In Section 2, we present the result in the scalar case in the context of an example. A suitably modified wave front tracking algorithm and the operator splitting method for the construction of solutions to the hyperbolic balance law (1.1) are outlined in Section 3. In Section 4, we prove the main theorems, in particular, we derive a sharp decay estimate on the *spreading of rarefaction waves* for general solutions of bounded variation to hyperbolic balance laws. Throughtout this paper, the emphasis is on the proof of Theorem 1.2 in the dissipative case.

2 Motivation

In this section, we study the scalar balance law as a motivation that gives rise to the results of the theorems.

Let $u = u(t,x)$ be a solution of the scalar balance law

$$u_t + \left(\frac{u^2}{2}\right)_x + \beta u = 0$$

with smooth initial data

$$u(0,x) = u_0(x) \ ,$$

where $\beta > 0$ constant. Define $w = w(t,x)$ as the solution of the Cauchy problem for the scalar Burgers's equation with source term

$$w_t + \left(\frac{w^2}{2}\right)_x + \beta w = 0$$

and initial data

$$w(0,x) = sgn(x) \cdot \sup_{|A|<2|x|} \int_A \frac{u_{0,x}(y)}{2}\, dy.$$

It should be noted that the above initial value problem corresponds to problem (1.17)-(1.18) with $Q = 0$. Now along each characteristic curve

$$\frac{dx}{dt} = u(t, x(t)),$$

the gradient satisfies

$$\frac{d}{dt} u_x(t, x(t)) = -(u_x)^2 - \beta u_x. \tag{2.1}$$

Note, when u_x is positive (2.1) yields that

$$\frac{d}{dt} u_x(t, x(t)) \leq 0,$$

which implies that $u_x(t, x(t))$ is a nonincreasing function.

Clearly, when $u_x(t, x(t))$ is positive, (2.1) implies that

$$u_x(t, x(t)) \leq \frac{e^{-\beta t}}{\left(\frac{1-e^{-\beta t}}{\beta}\right) + [u_{0x}(x(0))]^{-1}}. \tag{2.2}$$

By virtue of (1.19), for every $t > 0$ we have the relation

$$\mu^+ \preceq D_x w(t),$$

which includes the additional information (2.2). We remark that this relation is sharp in the sense that the reverse

$$D_x w(t) \preceq \mu_t^+$$

also holds, as long as no positive waves are cancelled by interacting with shocks.

3 Properties of hyperbolic balance laws and wave measures

In this section, we determine the solutions of the initial value problem

$$\partial_t u + \partial_x f(u) + g(u) = 0 \tag{3.1}$$
$$u(0,x) = u_0(x) \tag{3.2}$$

by constructing a family of Lipschitz semigroups P_t with some desired properties as the $s \to 0$ limit of an approximate sequence of operators P_t^s constructed by a suitably modified *wave front tracking algorithm*. In the following construction, the source term is handled by the *operator splitting* method.

Given a system of hyperbolic conservation laws, $g = 0$, with initial data of *small* total variation, existence and stability of solutions have been established in a series of articles. We refer the reader to the articles [7,16,9,10,6] and the references therein.

Roughly speaking, if we denote by $\mathcal{D}$ the domain

$$\mathcal{D}(\delta) = cl\{u \in \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n) : u \text{ is piecewise constant and } \Upsilon(u) < \delta\},$$

where the closure is taken in the $\mathbf{L}^1$-topology, then there exist $\delta_S, L_S > 0$ and a unique dynamical system (semigroup)

$$S : [0,+\infty] \times \mathcal{D}(\delta_S) \to \mathcal{D}(\delta_S)$$

satisfying a series of properties [7,16,9], such that if $u_0 \in \mathcal{D}$ is piecewise constant, then, for t sufficiently small, $u(\cdot,t) = S_t u_0$ coincides with the solution obtained by piecing together the standard entropy solutions of the Riemann problems determined by the jumps of u_0. Moreover, for each $\delta \leq \delta_S$ the domains $\mathcal{D}(\delta)$ are invariant under the evolution of the semigroup S.

A Lipschitz evolution operator P for the hyperbolic balance law (1.1) has been constructed in [1]. Under the diagonally dominance assumption, the evolution operator $P : [0,+\infty) \times \mathcal{D} \to \mathcal{D}$ turns out to be a semigroup defined on a suitable domain $\mathcal{D}$. Moreover, the evolution operator P satisfies the following

key estimate:

$$\|P_h u_0 - S_h u_0 + h \cdot g(u_0)\|_{L^1} = O(1)h^2 \quad \text{as} \quad h \to 0. \tag{3.3}$$

We state below the theorem which determines the global existence of solutions for the inhomogeneous system (3.1)–(3.2) as established in [1].

Theorem 3.1 *Under the dissipativeness assumption* (1.8)*, there exist a closed domain* $\mathcal{D} \subset \mathbf{L}^1(\mathbb{R}, \mathbb{R}^n) \cap \mathbf{BV}(\mathbb{R}, \mathbb{R}^n)$*, containing all* L^1*-functions with sufficiently small total variation, positive constants* β*,* C_0*,* $L_P > 0$ *and a unique semigroup* $P : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ *satisfying for each* $u_0, v_0 \in \mathcal{D}$ *and all* $s, t \geq 0$ *the following properties:*

1. $P_0 u_0 = u_0, \ P_{t+s} u_0 = P_t P_s u_0,$
2. $\|P_t u_0 - P_s v_0\|_{\mathbf{L}^1} \leq L_P(|s - t| + \|u_0 - v_0\|_{\mathbf{L}^1}),$
3. $\|P_h u_0 - S_h u_0 + h \cdot g(u_0)\|_{\mathbf{L}^1} = o(h)$ *as* $h \to 0$.
4. *the function* $u(x, t) = (P_t u_0)(x)$ *is a weak entropy solution of* (1.1) *with*

$$\textit{Tot. Var. } u(\cdot, t) \leq C_0 e^{-\beta t} \textit{Tot. Var. } u_0(\cdot) \tag{3.4}$$

5. *For all* $t \geq 0$*,* L^1*-stability of solutions holds:*

$$\|P_t u_0 - P_t v_0\|_{\mathbf{L}^1} \leq L_P e^{-\beta t} \|u_0 - v_0\|_{\mathbf{L}^1}. \tag{3.5}$$

On the other hand, assume that a Lipschitz map $u : [0, T] \to \mathbf{L}^1(\mathbb{R})$ *satisfies for all* $t \in [0, T) : u(t) \in \mathcal{D}$ *and*

$$\|u(t + h) - S_l u(t) + l \cdot g(u(t))\|_{\mathbf{L}^1} = o(l), \ \textit{as } l \to 0. \tag{3.6}$$

Then $u(t)$ *coincides with the flow trajectory* $P_t u(0)$.

It should be noted that the exponential decay factor is due to the dissipation induced by g.

In the case of a general source term $g = g(u)$, we have in general *blow up* of solutions. More presicely, there exist time $T > 0$ and a unique semigroup P defined on $[0, T) \times D \to D$ that satisfies the properties of the above theorem with the difference that estimates (3.4)–(3.5) become

$$\text{Tot. Var. } u(t, \cdot) \leq C_0 e^{\gamma t} \text{Tot. Var. } u_0(\cdot) \tag{3.7}$$

$$\|P_t u_0 - P_t v_0\|_{\mathbf{L}^1} \leq L_P e^{\gamma t} \|u_0 - v_0\|_{\mathbf{L}^1}. \tag{3.8}$$

where γ is a positive constant that depends on the source g. One can see that (3.7)–(3.8) imply possible blow up of solutions in finite time.

3.1 The Operator Splitting Method

In view of the discussion in the introduction, existence and stability of solutions to the balance law (1.1) are established in [17,2,13] using the three methods of construction. Here, we choose to present now the algorithm for the construction of solutions using a modified front-tracking method since the properties of the approximate solutions $u^{s,m}$ fit better in our analysis. We refer the reader to [16,2] for further details.

The algorithm is initiated, at time $t_0 = 0$, by approximating the initial data by a piecewise constant data u_0^s. For convenience, we drop the index s and denote this piecewise constant initial data by u_0 To pass from $t = t_k$ to $t = t_{k+1}$, we first solve approximately the ordinary differential equation

$$\partial_t u + g(u) = 0 \quad \text{on} \quad (t_k, t_{k+1})$$

and then solve independently, the conservation law

$$\partial_t u + \partial_x f(u) = 0.$$

More precisely the process is described below.

Let S be the semigroup for the homogeneous system and δ denote a fixed parameter $\delta \in (0, \varepsilon)$. Given a mesh time interval $\Delta t = s > 0$ and a function $u \in \mathcal{D}(\delta)$ the approximate operator P^s is defined as follows:

• At $t = 0$,

$$P_0^s u_0 = u_0$$

• For every $t \in (ks, (k+1)s)$, and for every $k \geq 0$, we solve the homogeneous problem by applying the semigroup operator S_{t-ks} to the solution P_{ks}^s, namely

$$P_t^s u_0 = S_{t-ks} P_{ks}^s u_0.$$

• To estimate the solution at time $t = ks$, $k \geq 1$, we add to the value of the solution $P_{ks-}^s u$ estimated as

$$P_{ks-}^s u_0 = \lim_{t \to ks-} P_t^s u_0 = S_s P_{(k-1)s}^s u_0,$$

the value of an approximate solution to the differential equation $\partial_t u + g(u) = 0$ evaluated at time $t = ks-$, namely

$$P_{ks}^s u_0 = \lim_{t \to ks-} P_t^s u_0 - s \cdot g(P_{ks-}^s u_0).$$

Using the above operating splitting method, given a mesh size $s = \Delta t$, an approximating sequence P_t^s is constructed for all times $t > 0$ and as $s \to 0$, P_t^s converges to the Lipschitz semigroup P associated with (3.1)–(3.2). Under the dissipativeness condition (1.8), the Lipschitz evolutionary operator P^s is an approximated semigroup defined for all $t \geq 0$ that satisfies

$$\begin{cases} P_t^s u_0 \in \mathcal{D}(\min(1, e^{2\beta s} e^{-2\beta t}) \cdot \delta), \\ \text{Tot.Var.} P_t^s u_0 \leq C_1 \min(1, e^{2\beta s} e^{-2\beta t}) \text{Tot. Var.} u_0, \\ \Upsilon(P_t^s u_0) \leq e^{-\beta t} \Upsilon(u_0) \end{cases} \tag{3.9}$$

for suitable positive constants C_1 and β. It should be noted that $0 < \beta < \tilde{\beta}$ and $\tilde{\beta} - \beta = O(\text{Tot. Var. } u_0)$.

Similarly, for a general source g, for any u within the domain $\mathcal{D}(\delta)$, the approximate trajectories $P_t^s u_0$ are defined within the interval $[0, T]$ and it holds

$$\begin{cases} P_s^t u_0 \in \mathcal{D}(e^{\gamma t} \delta), \\ \text{Tot. Var. } P_t^s u_0 \leq C_1 e^{\gamma t} \text{ Tot. Var. } u_0, \\ \Upsilon(P_t^s u_0) \leq e^{\gamma t} \Upsilon(u_0) \end{cases} \tag{3.10}$$

for suitable positive constants C_1 and γ.

So far, following [2], the above construction gives rise to a sequence of approximate solutions $u^s(\cdot, t) = P_t^s u_0$, $s = \Delta t$ with the property that as $s \to 0$, we can pass to the limit and u^s converges to the solution of balance laws (3.1)–(3.2). Now, the so called modified front-tracking approximate solution $u^{s,m}(x, t)$ can be obtained by approximating the semigroup S by the semigroup S^m associated with the front-tracking approximate solutions for the system of conservation laws ($g = 0$). Here, m denotes the front-tracking parameter and as $m \to \infty$, $u^m(\cdot, t) = S_t^m u_0$ converges to the entropy weak solution of the system of conservation laws ($g = 0$). For more details on the front-tracking method, we refer the reader to Bressan [7], Dafermos [16] and Holden-Risebro [20]. In other words, at every time step $(t_k, t_{k+1}) = (ks, (k+1)s)$, we obtain a piecewise constant function $u^{s,m}(t) = P_t^{s,m} u_0$ using the front-tracking approximate semigroups S^m, and at times $t = ks$, we apply the operating splitting method as described above on the modified operator $P^{s,m}$ at $t = ks-$. In this way, $u^{s,m}(t) = P_t^{s,m} u_0^s$ is a globally defined piecewise constant function with a finite number of discontinuities that enjoys all properties of a front tracking approximation and as $m \to \infty$, $u^{s,m}(t)$ converges to $u^s(t) = P_t^s u_0$. Thus, as $m \to \infty$ and $s \to 0$, we can extract a subsequence such that $u^{s,m} \to u$, where u is an entropy weak solution to balance laws (3.1)–(3.2). This sequence is employed in the proof of the theorems in Section 4. By the references mentioned above, it is clear that this construction is global in time for dissipative source g and it can be extended up to $t = T > 0$ for general source g.

Similarly, one could employ the random choice method of Glimm to approximate the semigroup S in each time step. This is precisely the idea in Dafermos–Hsiao [17]. In our analysis, we choose to employ the modified front-tracking sequence described above.

3.2 Wave measures

Here, we quote the main properties of the partial ordering. For a longer exposition, we refer to [11,7].

Let v be a bounded, nondecreasing function $v : \mathbb{R} \mapsto \mathbb{R}$ and $\mu = D_x v$ its distributional derivative. A positive Radon measure μ can be decomposed as the sum of a singular and an absolutely continuous part with respect to the Lebesgue measure, namely

$$\mu = \mu^{sing} + \mu^{ac}.$$

The absolutely continuous part corresponds to the usual derivative $z = v_x$, which is a nonnegative L^1 function defined almost everywhere.

Definition 3.2 *The symmetric rearrangement of z, denoted by $\hat{z}$ is defined to be the unique even function with the properties*

$$\hat{z}(x) = \hat{z}(-x), \quad \hat{z}(x) \geq \hat{z}(x') \quad \textit{if} \quad 0 < x < x', \tag{3.11}$$

$$\textit{meas}\,(\{x; \hat{z}(x) > c\}) = \textit{meas}\,(\{x; z(x) > c\}) \quad \textit{for every} \quad c > 0. \tag{3.12}$$

In accordance, the odd rearrangement of v, denoted by $\hat{z}$ is defined to be the unique function $\hat{v}$ with the property

$$\hat{v}(-x) = -\hat{v}(x), \quad \hat{v}(0+) = \frac{1}{2}\mu^{sing}(\mathbb{R}), \tag{3.13}$$

$$\hat{v}(x) = \hat{v}(0+) + \int_0^x z(y)\,dy \quad \textit{for } x > 0. \tag{3.14}$$

Hence $\hat{v}$ is convex for $x < 0$ and concave for $x > 0$.

The relation between the odd rearrangement $\hat{v}$ and the partial ordering $\preceq$ introduced earlier as well as further properties are expressed in the following lemma.

Lemma 3.3 *Let μ and μ' be two positive Radon measures. The following properties hold:*

(a) *If* $\mu = D_x v$, $\mu' = D_x v'$ *and* $\hat{v}$ *and* $\hat{v}'$ *denote the odd rearrangements of* v, v' *respectively, then*

$$\mu \preceq D_x \hat{v} \preceq \mu.$$

Moreover,

$$\hat{v}(x) = sgn(x) \cdot \sup_{|A| \leq 2|x|} \frac{\mu(A)}{2}, \tag{3.15}$$

and

$$\mu \preceq \mu' \quad \text{if and only if} \quad \hat{v}(x) \leq \hat{v}'(x) \ \text{ for all } x > 0. \tag{3.16}$$

(b) *Consider any finite partition* $\mathbb{R} = J_1 \cup \cdots \cup J_N$. *Let us denote the restrictions of* μ, μ' *to the set* J_k, $k = 1, \ldots, N$ *by*

$$(\mu\Big\rfloor_{J_k})(A) = \mu(A \cap J_k), \quad (\mu'\Big\rfloor_{J_k})(A) = \mu'(A \cap J_k).$$

Then,

$$\text{if} \quad \mu(A\Big\rfloor_{J_k}) \preceq \mu'(A\Big\rfloor_{J_k}) \quad \text{for each} \quad J_k \quad \text{then} \quad \mu \preceq \mu'.$$

(c) *Assume that* $\mu \preceq D_s w$ *for some nondecreasing odd function* w.

$$\text{If} \quad |\mu^{\#} - \mu|(\mathbb{R}) \leq \varepsilon, \quad \text{then} \quad \mu^{\#} \preceq D_s \left[w + sgn(s) \cdot \frac{\varepsilon}{2} \right].$$

Proof. We refer the reader to [11] for the proof. □

The next proposition yields the lower semicontinuity of the partial ordering $\preceq$ and it will be crucial in obtaining our main result.

Proposition 3.4 *The following statements hold true:*

(a) *Consider a sequence of measures* μ_ν *converging weakly to a measure* μ. *Assume that the positive parts satisfy* $\mu_\nu^+ \preceq D w_\nu$ *for some nondecreasing odd functions* $s \mapsto w_\nu(s)$, *concave for* $s > 0$. *Let* w *be the odd function such that*

$$w(s) = \liminf_{\nu \to \infty} w_\nu(s) \quad \text{for} \quad s > 0.$$

Then the positive part of μ *satisfies*

$$\mu^+ \preceq D_s w. \tag{3.17}$$

(b) *Consider a sequence of functions* $u_{\nu,s}$ *with uniformly small total variation and call* μ_ν^{i+} *the corresponding measures of positive* i*-waves. Let* $s \mapsto w_{\nu(s)}$, $\nu \geq 1$, *be a sequence of odd, nondecreasing functions, concave for* $s > 0$, *such that*

$$\mu_\nu^{i+} \preceq D_s \left[w_\nu + C_0 sgn(s)(Q_0 - Q(u_\nu)) \right] \tag{3.18}$$

for some Q_0. Assume that $u_\nu \to u$ and $w_\nu \to w$ in L^1_{loc}. Then the measure of positive i-waves in u satisfies

$$\mu^{i+} \preceq D_s \left[w + C_0 sgn(s)(Q_0 - Q(u)) \right] . \tag{3.19}$$

Proof: The proof follows the line of argument presented in [5], [11]. □

4 Proof of the main theorems

In this section, we present the proof of Theorems 1.2 and 1.3. We focus on the proof of Theorem 1.2 with an extensive and detailed analysis in Subsections 4.1 and 4.2. The presence of the source g, under the diagonally dominance condition, brings additional challenges, therefore, new delicate estimates need to be derived. We present these estimates in great detail. First, we establish a version of the results in Subsection 4.1 in the context of a prototype hyperbolic balance law and then we use them in an iteration scheme in the proof of the theorem in Subsection 4.2. The proof of Theorem 1.3, for a general source g, follows easily and it can be found in Subsection 4.3.

4.1 A "simplified" model

The aim in this subsection is to obtain a spreading estimate for a "prototype" balance law presented in Lemma 4.1. This decay estimate will be of use in an approximation argument given in the sequel in the proof of Theorem 1.2.

Lemma 4.1 *Let u be an entropy weak solution of the hyperbolic system of balance laws* (1.1) *and* (1.5) *of small variation under hypothesis* (1.8) *and μ_t^{i+} be the corresponding measure of its positive i–waves at time t.*

Consider the solution $w : [0, \tau) \times \mathbb{R} \to \mathbb{R}$ to Burgers's equation with source

$$\partial_t w + \partial_x (\frac{w^2}{2}) + \beta w = 0 \ , \tag{4.1}$$

and initial data

$$w(0, x) = sgn(x) \cdot \sup_{|A| \le 2|x|} \frac{\mu_0^{i+}(A)}{2} \ , \tag{4.2}$$

where $\beta > 0$ is given in Theorem 3.1. For $t = \tau$, set

$$w(\tau, x) = w(\tau-, x) + \kappa sgn(x) \cdot [Q(u_0) - Q(u(\tau))], \tag{4.3}$$

then the following estimate holds

$$\mu_\tau^{i+} \preceq D_x w(\tau). \tag{4.4}$$

Our analysis follows similar line of argument to the one presented in Bressan–Yang [11] in the context of hyperbolic conservation laws ($g \equiv 0$). New delicate estimates are presented here in order to deal with the challenges that the presence of the source term $g = g(u)$ involves. In order to pass from the hyperbolic conservation laws ($g \equiv 0$) to the balance laws ($g \neq 0$), important modifications in these techniques need to be made for the treatment of the source g. For the benefit of the reader, the presentation of the proof has an emphasis on the role of the source g.

Before we prove this lemma, we introduce quantities that approximate the distributional derivative $D_x w$ of (4.3) and the measure of positive i–waves μ^{i+} of u over subintervals that are bounded by i–characteristics. We prove an important relation between these quantities in Proposition 4.2 that would lead the proof of the above lemma in the end of the subsection.

We assume that the initial data u_0 to (1.1) has the following property: There exists a finite number of points

$$\bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_r$$

such that $u_0(x)$ is constant for $x < \bar{x}_1$ and $x > \bar{x}_r$, smooth away from these points $\bar{x}_l$ and its derivative component $l_i(u)u_x$ is constant on each interval $(\bar{x}_l, \bar{x}_{l+1})$. Moreover, we assume that the Glimm interaction potential

$$t \mapsto Q(u(t)) \tag{4.5}$$

is continuous at time $t = \tau$.

Let $\varepsilon > 0$ be a fixed small positive parameter. By the assumption on the initial data, we can approximate the positive measure μ_τ^{i+} of i–waves by a piecewise constant function $\psi^\tau(x)$ with jumps at finitely many points

$$x_1(\tau) < x_2(\tau) < \cdots < x_N(\tau),$$

i.e. there are contants ψ_j^τ, $j = 1, \ldots, N$

$$\psi^\tau(x) = \psi_j^\tau, \quad \text{for} \quad x_j(\tau) < x < x_{j+1}(\tau),$$

such that the integral of $\psi^\tau(\cdot)$ is equal to the measure of positive i–waves over each subinterval $[x_j(\tau), x_{j+1}(\tau))$ and it is a good approximation over the whole real line. Namely,

$$\mu_\tau^{i+}(I_j(\tau)) = \psi_j^\tau(x_{j+1}(\tau) - x_j(\tau))$$

and

$$\int \left| \frac{d\mu_\tau^{i+}}{dx} - \psi^\tau(x) \right| dx < \varepsilon \, .$$

Moreover, we consider i–characteristics $t \to x_j(t)$, $t \in [0, \tau]$ that emanate from the points $(x_j(\tau), \tau)$ of discontinuity of $\psi^\tau(\cdot)$. We set $\psi^0(x) = l_i(u) u_x(x, 0)$, such that either $x_j(0) = x_{j+1}(0)$ or else $\psi^0(x)$ is constant on $(x_j(0), x_{j+1}(0))$ and we denote its constant value by ψ_j^0. By convention, we assume that $(\psi_j^0)^{-1} = 0$ when $x_j(0) = x_{j+1}(0)$.

For every $t \in [0, \tau]$ and $j = 1, \ldots, N$, we consider the interval

$$I_j(t) \doteq [x_j(t), x_{j+1}(t)) \, , \tag{4.6}$$

of length

$$s_j(t) \doteq x_{j+1}(t) - x_j(t) \, , \tag{4.7}$$

and the region

$$\Delta_j^{[0,t]} \doteq \{(s, x) : \, s \in [0, t], \, , x \in I_j(s)\} \, . \tag{4.8}$$

Now, for each j we denote by $\Gamma_j^{[0,t]}$ the total amount of wave interaction that took place within the region $\Delta_j^{[0,t]}$. We can view this quantity as the difference between the interaction potential at the initial time $t = 0$ and a later time t, that is,

$$\Gamma_j^{[0,t]} = Q_j(u(0)) - Q_j(u(t)) \, ,$$

where $Q_j(u(\cdot))$ denotes the interaction potential in u along the interval $I_j(\cdot)$ as defined in (1.16).

Then, the measures of positive i–waves over the interval $I_j(t)$ at time $t = \tau$ and $t = 0$ satisfy

$$\sigma_j^\tau \doteq \mu_\tau^{i+}(I_j(\tau)) = \psi_j^\tau \left(x_{j+1}(\tau) - x_j(\tau) \right) \, , \tag{4.9}$$

$$\sigma_j^0 \doteq \lim_{t \to 0+} \mu_\tau^{i+}(I_j(t)) \, . \tag{4.10}$$

The goal is to compare the measure μ_τ^{i+} of positive i–waves in u with the distributional derivative of the solution to (4.3) over the interval $I_j(\tau)$. To achieve this, we define a nondecreasing function $\zeta \mapsto w_j(\zeta)$

$$w_j(\zeta) \doteq \min \left\{ e^{-\beta\tau} \sigma_j^0, \; \frac{e^{-\beta\tau} \zeta}{(\psi_j^0)^{-1} + \frac{1}{\beta}(1 - e^{-\beta\tau})} \right\} + \kappa e^{-\beta\tau} sgn(\zeta) \, \Gamma_j^{[0,\tau]} \, , \tag{4.11}$$

which corresponds to (4.1)-(4.3). Note that the second term in parenthesis corresponds to the solution of Burgers's with source $\beta \, w$ and having initial data ψ_j^0. The last term is due to the amount of interaction that is included in

(4.3). It should be noted that the above function reduces to the one considered in Bressan–Yang [11] as g tends to 0 uniformly.

In addition, we define a nondecreasing odd function $\zeta \mapsto \eta_j(\zeta)$ as follows

$$\eta_j(\zeta) = \psi_j^\tau \, \min\{\zeta, \, [x_{j+1}(\tau) - x_j(\tau)]\} \, , \qquad \zeta > 0 \, . \tag{4.12}$$

Note that $\eta_j(0) = 0$ and $\eta_j(\zeta)$ is constant for $\zeta \geq s_j(\tau)$. By (4.9), one can see the relation between η_j and $\mu^{i+}(I_j(\tau))$. Figure 1 shows the graphs of both w_j and η_j for $\zeta > 0$.

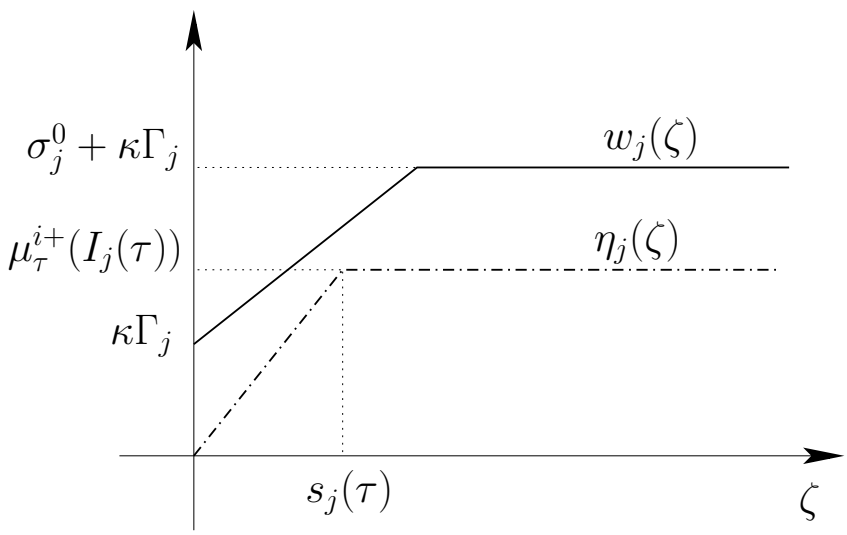

Fig. 1. The graphs of w_j and η_j

As mentioned above, the two quantities that would be studied to prove estimate (4.4) in Lemma 4.1 are w_j and η_j. The key estimate is given in the next proposition.

Proposition 4.2 *For each $j = 1, \ldots, N$, consider the functions w_j and η_j as defined in* (4.11) *and* (4.12)*, respectively. Then for every $\zeta > 0$,*

$$\eta_j(\zeta) \leq w_j(\zeta) \, . \tag{4.13}$$

Proof. It is easy to show that the result holds when $\sigma_j^0 = 0$. Indeed, in such a case, (1.16) and (3.9) imply that there is a positive constant c so that

$$\eta_j(\zeta) \leq \mu_\tau^{i+}(I_j(\tau)) \leq c\,\Gamma_j^{[0,\tau]}, \qquad \text{for all } \; \zeta > 0 \, . \tag{4.14}$$

Choosing $\kappa > c$, (4.11) yields

$$w_j(\zeta) = \kappa\Gamma_j^{[0,\tau]} \geq c\,\Gamma_j^{[0,\tau]} \, , \tag{4.15}$$

which implies the result.

Now, lets consider the case $\sigma_j^0 > 0$. Without loss of generality, we can assume that the length of the interval $I_j(0)$ is positive, namely $x_j(0) < x_{j+1}(0)$. Otherwise, one could treat the case $x_j(0) = x_{j+1}(0)$ in the limit $x_{j+1}(0) - x_j(0) \to 0$.

By account of (1.16) and (3.9), there is a positive constant c such that

$$\mu_\tau^{i+}(I_j(\tau)) \leq e^{-\beta\tau}\sigma_j^0 + c\, e^{-\beta\tau}\Gamma_j^{[0,\tau]} \tag{4.16}$$

over the region $\Delta_j^{[0,\tau]}$. It should be noted that the exponential decay factor in (4.16) is due to the dissipativeness assumption on g. If the amount of interaction is dominant in comparison to the initial amount of i–waves, namely

$$\Gamma_j^{[0,\tau]} \geq \frac{1}{6c}\sigma_j^0, \tag{4.17}$$

then it yields

$$\eta_j(\zeta) \leq \mu_\tau^{i+}(I_j(\tau)) \leq 7ce^{-\beta\tau}\Gamma_j^{[0,\tau]} \tag{4.18}$$

and estimate (4.13) holds for $\kappa > 7c$.

It remains to investigate the opposite case, when the initial amount σ_j^0 is dominant to $\Gamma_j^{[0,\tau]}$, i.e.

$$\Gamma_j^{[0,\tau]} < \frac{1}{6c}\sigma_j^0. \tag{4.19}$$

Recall that

$$\sigma_j^0 = (x_{j+1}(0) - x_j(0))\psi_j^0 > 0\ . \tag{4.20}$$

and

$$\sigma_j^\tau = s_j(\tau)\psi_j^\tau\ . \tag{4.21}$$

To prove (4.13), it suffices to establish it for the special value $\zeta = s_j(\tau)$, see Figure 1. For this value of ζ, (4.13) reduces to

$$\eta_j(s_j(\tau)) = \sigma_j^\tau \leq \min\left\{e^{-\beta\tau}\sigma_j^0,\ \frac{e^{-\beta\tau}s_j(\tau)}{(\psi_j^0)^{-1} + \dfrac{1-e^{-\beta\tau}}{\beta}}\right\} + e^{-\beta\tau}\kappa\Gamma_j^{[0,\tau]} = w_j(s_j(\tau))\ . \tag{4.22}$$

By virtue of (4.16), it suffices to verify that there is a constant $\kappa > 0$ so that

$$\sigma_j^\tau \leq \frac{e^{-\beta\tau}s_j(\tau)}{(\psi_j^0)^{-1} + \dfrac{1-e^{-\beta\tau}}{\beta}} + e^{-\beta\tau}\kappa\Gamma_j^{[0,\tau]}\ , \tag{4.23}$$

which can be deduced by

$$s_j(0) + \sigma_j^0\left(\frac{1-e^{-\beta\tau}}{\beta}\right) - c_1\left(\frac{1-e^{-\beta\tau}}{\beta} + (\psi_j^0)^{-1}\right)\Gamma_j^{[0,\tau]} \leq s_j(\tau) \tag{4.24}$$

for some positive constant c_1 using estimate (4.16).

Thus, from here and on, the goal is to establish estimate (4.24). Let $\{u^{s,m}\}_{s,m}$ be a sequence of front-tracking approximate solutions of the solution $u(t,x)$ to system of balance laws (1.1) that can be constructed as described in Subsection 3.2. Hence as $m \to \infty$ and $s \to 0$, $u^{s,m}$ converges to u in L^1_{loc}. Moreover, the approximate initial data $u^{s,m}(x,0)$ contains exactly m rarefaction fronts that are equally spaced along the interval $I_j(0)$ and each initial front has strength $\sigma_k(0) = \sigma_j^0/m$ for $k = 1, \dots, m$. For the convenience of the reader, we drop the index s that corresponds to the time step $s = \Delta t$ and denote this sequence by u^m.

We fix $j = 1, \dots, N$ and decompose the interval $I_j(t) = [x_j(t), x_{j+1}(t))$ into subintervals $J_k(t) = [y_k(t), y_{k+1}(t))$ with the property that at time t a front of strength $\sigma_k(t)$ is located at $x = y_k(t)$.

For each $k = 1, \dots, m$, we denote the length of the interval $J_k(t)$ by

$$l_k(t) \doteq y_{k+1}(t) - y_k(t) \ , \tag{4.25}$$

consider the region

$$\Delta_{k,j}^{[0,t]} \doteq \{(s,x); \quad s \in [0,t], \quad x \in J_k(s)\} \tag{4.26}$$

and let $\Gamma_{k,j}^{[0,t]}$ be the amount of interaction of u^m within $\Delta_{k,j}^{[0,t]}$. Then for all $t < \tau$, we have $\Gamma_{k,j}^{[0,t]} \leq \Gamma_{k,j}^{[0,\tau]}$.

Following [11], we consider a set of indices $\mathcal{K} \subset \{1, \dots, m\}$ with the property that $k \in \mathcal{K}$ if

$$\sigma_k(0) < 5c\Gamma_{k,j}^{[0,\tau]} \ . \tag{4.27}$$

Observe that if k, $k+1 \notin \mathcal{K}$, then $J_k(t) = [y_k(t), y_{k+1}(t))$ satisfies

$$V_k(t) \leq \frac{d}{dt} l_k(t) + c_1 \sum_{\nu \in \mathcal{A}_k(t)} |\sigma_\nu(t)| \ , \tag{4.28}$$

for some positive constant c_1. Here, $V_k(t)$ is the total amount of i–waves in $J_k(t)$ and $\mathcal{A}_k(t)$ corresponds to the set of all wave fronts of different families that are crossing $J_k(t)$. However, because of the dissipation, there is a positive constant β, so it holds

$$e^{\beta t} V_k(t) \geq \sigma_k(0) - c\Gamma_{k,j}^{[0,\tau]} \ . \tag{4.29}$$

Integrating (4.28) and using (4.29), we get

$$l_k(\tau) \geq l_k(0) + \frac{1 - e^{-\beta\tau}}{\beta}(\sigma_k(0) - c\Gamma_{k,j}^{[0,\tau]}) - c_1 \int_0^\tau \sum_{\nu \in \mathcal{A}_k(t)} |\sigma_\nu(t)| \, dt \ . \tag{4.30}$$

Now, we need to control the last term, i.e. the integral of the strengths $|\sigma_\nu|$ of the waves of different families at time t. These wave fronts are of three possible

types: (i) $\sigma_{\nu,\alpha}$: wave fronts that are located in J_k by entering the curves y_k or y_{k+1}; (ii) $\sigma_{\nu,\beta}$: wave fronts that are generated through wave interactions within $\Delta_{k,j}^{[0,t]}$ and (iii) $\sigma_{\nu,\gamma}$: wave fronts that are located in $J_k(0)$.

Case (i): Every wave front $\sigma_{\nu,\alpha}$ of different family remains within $\Delta_{k,j}^{[0,t]}$ for a time length proportional to $l_k(t)$. By account of (4.27), since k, $k+1 \notin \mathcal{K}$, it follows $\sigma_k(t) \geq \sigma_k(0)/2$. Then using the total amount of interaction $\Gamma_{k,j}^{[0,t]}$ and $\sigma_k(0) = \sigma_j^0/m$, it yields

$$\sum_{\nu} |\sigma_{\nu,\alpha}| \leq \frac{2m}{\sigma_j^0} \Gamma_{k,j}^{[0,t]} \ . \tag{4.31}$$

Case (ii): By (3.9), wave fronts $\sigma_{\nu,\beta}$ that are generated within $\Delta_{k,j}^{[0,t]}$ satisfy

$$\sum_{\nu} |\sigma_{\nu,\beta}| \leq c\, e^{-\beta t} \Gamma_{k,j}^{[0,t]} \ . \tag{4.32}$$

Case (iii): Similarly, wave fronts $\sigma_{\nu,\gamma}$ that are in $J_k(0)$ satisfy

$$\sum_{\nu} |\sigma_{\nu,\gamma}| \leq \sum_{\nu \in \mathcal{A}_k(0)} |\sigma_\nu(0)| + c\, e^{-\beta t} \Gamma_{k,j}^{[0,t]} \tag{4.33}$$

and initially these fronts are equally spaced along $I_j(0)$.

Recalling the property

$$\Gamma_{k,j}^{[0,t]} \leq \Gamma_{k,j}^{[0,\tau]} \ , \quad \text{for } t \leq \tau$$

and using estimates (4.31)–(4.33), we obtain the bound

$$\begin{aligned} \int_0^\tau \sum_{\nu \in \mathcal{A}_k(t)} |\sigma_\nu(t)|\, dt \leq & O(1) \max_{t \in [0,\tau]} l_k(t) \frac{2m}{\sigma_j^0} \Gamma_{k,j}^{[0,\tau]} + O(1) \int_0^\tau e^{-\beta t}\, dt\, \Gamma_{k,j}^{[0,\tau]} \\ & + O(1) \frac{1}{m} (s_j(0)+1) \sum_{\nu \in \mathcal{A}_k(0)} |\sigma_\nu(0)| \ . \end{aligned} \tag{4.34}$$

Combining with (4.30), we arrive at

$$\begin{aligned} l_k(\tau) \geq & l_k(0) + \frac{1-e^{-\beta\tau}}{\beta} (\sigma_k(0) - O(1)\Gamma_{k,j}^{[0,\tau]}) - O(1) \frac{2m}{\sigma_j^0} \Gamma_{k,j}^{[0,\tau]} \max_{t \in [0,\tau]} l_k(t) \\ & - O(1) \frac{1}{m} (s_j(0)+1) \sum_{\nu \in \mathcal{A}_k(0)} |\sigma_\nu(0)| \ . \end{aligned} \tag{4.35}$$

Next, we need to control $\max_{t \in [0,\tau]} l_k(t)$ in the above expression and we claim that

$$\max_{t \in [0,\tau]} l_k(t) \leq 2 l_k(\tau) \ . \tag{4.36}$$

To prove the claim, we can follow similar line of argument as in Bressan–Yang [11], where the case $g = 0$ was treated. We omit the proof here and refer the

reader to pp. 672–673 of [11] . Further details of the computations can be found in pp. 213 of [7].

Since $\sigma_k(0) = \sigma_j^0/m$, inequality (4.35) can be written as

$$
\begin{aligned}
l_k(\tau) \geq & \frac{l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{2m}{\sigma_j^0}}{1 + O(1)\frac{m}{\sigma_j^0}\Gamma_{k,j}^{[0,\tau]}} - O(1)\frac{1}{\beta}(1-e^{-\beta\tau})\Gamma_{k,j}^{[0,\tau]} \\
& - O(1)\frac{1}{m}(s_j(0)+1)\sum_{\nu\in\mathcal{A}_k(0)}|\sigma_\nu(0)| \,. \qquad (4.37)
\end{aligned}
$$

Using (4.37), it yields

$$
\begin{aligned}
s_j(\tau) = & \sum_{k=1}^{m} s_k(\tau) \geq \sum_{k\notin\mathcal{K}} s_k(\tau) \\
\geq & \sum_{k\notin\mathcal{K}} \Bigg[\frac{l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{\sigma_j^0}{m}}{1 + O(1)\frac{m}{\sigma_j^0}\Gamma_{k,j}^{[0,\tau]}} - O(1)\frac{1}{\beta}(1-e^{-\beta\tau})\Gamma_{k,j}^{[0,\tau]} \\
& - O(1)\frac{1}{m}(s_j(0)+1)\sum_{\nu\in\mathcal{A}_k(0)}|\sigma_\nu(0)| \Bigg] \,. \qquad (4.38)
\end{aligned}
$$

Because $\sum_{k\notin\mathcal{K}}\Gamma_{k,j}^{[0,\tau]} \leq \Gamma_j^{[0,\tau]}$ and the total variation of the initial data is bounded, estimate (4.38) reduces to

$$
\begin{aligned}
s_j(\tau) \geq & \sum_{k\notin\mathcal{K}} \left[(l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{\sigma_j^0}{m})(1 - O(1)\frac{m}{\sigma_j^0}\Gamma_{k,j}^{[0,\tau]}) \right] - O(1)\frac{1}{\beta}(1-e^{-\beta\tau})\Gamma_j^{[0,\tau]} \\
& - O(1)\frac{1}{m}(s_j(0)+1) \\
\geq & \sum_{k\notin\mathcal{K}} \left[l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{\sigma_j^0}{m} \right] - O(1)\frac{s_j(0)}{\sigma_j^0}\Gamma_j^{[0,\tau]} - O(1)\frac{1}{\beta}(1-e^{-\beta\tau})\Gamma_j^{[0,\tau]} \\
& - O(1)\frac{1}{m}(s_j(0)+1) \,, \qquad (4.39)
\end{aligned}
$$

where we used the fact that the intervals $J_k(0)$ are of equal length.

Now by virtue of definition (4.27), we get

$$
|\mathcal{K}| \leq 5c\,\frac{m}{\sigma_j^0}\Gamma_j^{[0,\tau]} \,, \qquad (4.40)
$$

where $|\cdot|$ denotes the cardinality of the set. Using $l_k(0) = s_j(0)/m$ for all $k = 1, \ldots, m$ and (4.40), we have

$$\begin{aligned}
&\sum_{k\notin\mathcal{K}}[l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{\sigma_j^0}{m}] \\
&\qquad = \sum_{k=1}^{m}[l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{\sigma_j^0}{m}] - \sum_{k\in\mathcal{K}}[l_k(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\frac{\sigma_j^0}{m}] \\
&\qquad = (s_j(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\sigma_j^0)(1-|\mathcal{K}|/m) \\
&\qquad \geq s_j(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\sigma_j^0 - 5c\left(\frac{s_j(0)}{\sigma_j^0} + \frac{1}{\beta}(1-e^{-\beta\tau})\right)\Gamma_j^{[0,\tau]} . \quad (4.41)
\end{aligned}$$

Combining (4.39) and (4.41), we arrive at

$$\begin{aligned}
s_j(\tau) \geq & s_j(0) + \frac{1}{\beta}(1-e^{-\beta\tau})\sigma_j^0 - O(1)\left(\frac{s_j(0)}{\sigma_j^0} + \frac{1}{\beta}(1-e^{-\beta\tau})\right)\Gamma_j^{[0,\tau]} \\
& - O(1)\frac{1}{m}(s_j(0)+1) . \quad (4.42)
\end{aligned}$$

Letting $m \to \infty$ and $s = \Delta t \to 0$, we prove (4.24). In view of the analysis in the begining of the proof, the result follows. □

By account of estimate (4.13), we now prove Lemma 4.1.

Proof of Lemma 4.1. Consider a sequence of initial data $\{u_0^r\}$ that converges to the initial data u_0 of (1.1) as $r \to \infty$ with the properties: there exists a finite number of points

$$\bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_r$$

such that u_0^r is constant for $x < \bar{x}_1$ and $x > \bar{x}_r$, smooth away from these points $\bar{x}_l$ and the i–measure $\mu_i(u_0^r)$ is constant on each interval $(\bar{x}_l, \bar{x}_{l+1})$ for $l = 0, \ldots, r-1$. Moreover, as $r \to \infty$,

$$Q(u_0^r) \to Q(u_0), \qquad \mu_0^{r,i+} \doteq \mu^{i+}(u_0^r) \to \mu^{i+}(u_0) . \quad (4.43)$$

Denote by $u_r(t,x)$ the solution to (1.1) with initial data u_0^r and $\mu_t^{r,i+}$ the corresponding measure of positive waves. Let $w^r(t,x)$ be the solution to (4.1) with initial data

$$w^r(0,x) = sgn(x) \cdot \sup_{|A|\leq 2|x|} \frac{\mu_0^{r,i+}(A)}{2} . \quad (4.44)$$

for $t < \tau$. For $t = \tau$, set

$$w^r(\tau,x) = w^r(\tau-,x) + \kappa sgn(x) \cdot [Q(u_0^r) - Q(u_r(\tau))] , \quad (4.45)$$

where κ is the positive constant given in (4.11).

In view of the analysis before Proposition 4.2, we can define nondecreasing functions $w_j^r(\cdot)$ and $\eta_j^r(\cdot)$ using (4.11) and (4.12), respectively, since the initial data u_0^r satisfy the assumptions imposed. By Proposition 4.2, it follows

$$\eta_j^r(\zeta) \leq w_j^r(\zeta) \quad \text{for all } \zeta > 0 \,. \tag{4.46}$$

By definition of $\eta_j^r(\zeta)$, we have

$$\sup_{|A|\leq 2|\zeta|} \frac{\mu_\tau^{r,i+}(A \cap I_j(\tau))}{2} \leq \eta_j^r(\zeta) + \varepsilon_j \,, \tag{4.47}$$

where

$$\sum_j \varepsilon_j < \varepsilon \,. \tag{4.48}$$

On the other hand, by (4.11) we get

$$\begin{aligned} w_j^r(\zeta) \leq & \frac{e^{-\beta\tau}\zeta}{(\psi_j^0)^{-1} + \frac{1}{\beta}(1-e^{-\beta\tau})} + \kappa e^{-\beta\tau} sgn(\zeta)\, \Gamma_j^{[0,\tau]} \\ \leq & w^r(\tau,\zeta) \qquad \text{on } \zeta \in I_j(\tau) \end{aligned} \tag{4.49}$$

Combining (4.47) and (4.49), we arrive at

$$\mu_{r,\tau}^{i+}\Big|_{I_j} \preceq D_\zeta w^r(\tau)\Big|_{I_j} \tag{4.50}$$

for all j. Since this relation holds over every interval I_j, by Lemma 3.3 part (b), it follows that it holds on $\mathbb{R}$. By the lower semicontinuity property, (cf. Proposition 3.4), we get

$$\mu_{r,\tau}^{i+} \preceq D_x w^r(\tau) = D_x[w^r(\tau-,x) + \kappa\, sgn(x)(Q(u_0^r) - Q(u_r(\tau)) + \mathcal{O}(1)\varepsilon] \,. \tag{4.51}$$

Note that as $r \to \infty$, $w^r(t)$ converges to the solution $w(t)$ of (4.1)–(4.2) for $t < \tau$. By (4.45), it yields

$$w(\tau,x) = w(\tau-,x) + \kappa sgn(x) \cdot [Q(u_0) - Q(u(\tau))] \,. \tag{4.52}$$

Then by letting $\varepsilon \to 0$ and $r \to \infty$ in (4.51) and using the lower semicontinuity property again, we deduce

$$\mu_\tau^{i+} \preceq D_x w(\tau) = D_x[w(\tau-,x) + \kappa\, sgn(x)[Q(u_0^r) - Q(u(\tau))] \,. \tag{4.53}$$

The proof of the lemma is complete. □

4.2 Proof of Theorem 1.2

Consider the system of balance laws

$$\partial_t u + \partial_x f(u) + g(u) = 0 \tag{4.54}$$

and let μ_t^{i+} be the measure of positive i–measure in $u(\cdot,t)$ as defined in Subsection 1.1. We will prove the theorem in the dissipative case, i.e. when the source $g(u)$ satisfies condition (1.8). By Theorem 3.1, there exists a global in time solution $u(\cdot,t)$ that decays exponentially fast to the equilibrium as $t\to\infty$.

Fix $\tau>0$. The solution w to (1.17)–(1.18) on $[0,\tau]$ can be constructed as a limit of a sequence of approximate solutions w_m as follows: Consider a sequence of partitions

$$0=t_0^m<t_1^m<\cdots<t_{N_m}^m=\tau \tag{4.55}$$

of $[0,\tau]$ so that $|t_{k+1}^m-t_k^m|\to 0$ as $m\to\infty$ uniformly in $k=0,\dots N_m-1$. Fix m, we now define the approximate solution $w_m(t,x)$ on $[0,\tau]$ by induction: First, take

$$w_m(0,x)=sgn(x)\cdot\sup_{|A|\le 2|x|}\frac{\mu_0^{i+}(A)}{2}\ . \tag{4.56}$$

Then, let $w_m(t,\cdot)$ be the solution to (4.1) on $[0,t_1^m)$ with data (4.56) and set

$$w(t_1^m,x)=w(t_1^m-,x)+\kappa\mathrm{sgn}(x)[Q(u(0))-Q(u(t_1^m))]\ . \tag{4.57}$$

Suppose that $w_m(t,\cdot)$ is defined for all $t\le t_{k-1}$ for $k=1,\dots,N_m$. Then, we define $w_m(t,\cdot)$ on $[t_{k-1}^m,t_k^m)$ to be the solution of (4.1) with initial data

$$w(t_{k-1}^m,x)=w(t_{k-1}^m-,x)+\kappa\mathrm{sgn}(x)[Q(u(t_{k-2}^m))-Q(u(t_{k-1}^m))] \tag{4.58}$$

and set

$$w(t_k^m,x)=w(t_k^m-,x)+\kappa\mathrm{sgn}(x)[Q(u(t_{k-1}^m))-Q(u(t_k^m))]\ . \tag{4.59}$$

By induction, we obtain $w_m(t,\cdot)$ for all $t\in[0,\tau]$. Now, we claim that the sequence $\{w_m\}$ converges to a unique limit w that satisfies the desired system (1.17)–(1.18). To prove this claim, let φ be a test function on $[0,\tau]\times(-\infty,\infty)$, then by (4.1) and (4.59), we get

$$\begin{aligned}
\int_0^\tau\int\varphi_t w+\varphi_x\frac{w^2}{2}-\varphi\beta w\,dx\,dt &= \lim_{m\to\infty}\sum_{k=1}^{N_m}\int_{t_{k-1}^m}^{t_k^m}\int\varphi_t w_m+\varphi_x\frac{w_m^2}{2}-\varphi\beta w_m\,dxdt\\
&=-\lim_{m\to\infty}\sum_k\int_{t_{k-1}^m}^{t_k^m}\int\varphi(t,x)\left(\partial_t w_m+\partial_x\frac{w_m^2}{2}+\beta w_m\right)dxdt\\
&\quad+\lim_{m\to\infty}\sum_k\int\varphi(t_k^m,x)w_m(t_k^m-)-\varphi(t_{k-1}^m,x)w_m(t_{k-1}^m+)\,dx\\
&=\lim_{m\to\infty}\sum_k\int\kappa\,\varphi(t_k^m,x)\left(Q(u(t_k^m))-Q(u(t_{k-1}^m))\right)dx\\
&=\int_0^\tau\int\kappa\,\varphi(t,x)\,\frac{dQ(u(t))}{dt}\,dx\,dt
\end{aligned} \tag{4.60}$$

Thus, the limit w satisfies system (1.17)–(1.18).

It should be noted that the flow given by (4.1) is positively invariant on the set of nondecreasing odd functions ϕ that are concave for $x > 0$ and it is also order preserving, i.e. if ϕ and ϕ' are solutions to (4.1), then for every $t > 0$,

$$D_x\phi(0) \preceq D_x\phi'(0) \quad \text{implies} \quad D_x\phi(t) \preceq D_x\phi'(t) \ .$$

Since the sequence w_m consists of nondecreasing odd functions, concave for $x > 0$ constructed via the flow (4.1), we apply Lemma 4.1 on each interval $[t_k^m, t_{k+1}^m]$ and use the above properties to get

$$\mu_\tau^{i+} \preceq D_x w_m(\tau).$$

As $m \to \infty$, the result of Theorem 1.2 in the dissipative case follows. □

4.3 Proof of Theorem 1.3

In view of the discussion in Section 3, when assumption (1.8) does not hold necessarily, then the entropy weak solution $u(\cdot, t)$ to

$$\partial_t u + \partial_x f(u) + g(u) = 0$$

exists locally in time. Let $T > 0$ be the maximal time of existence and fix $\tau \in [0, T]$. Then the result of Lemma 4.1 holds when (4.1) is replaced by

$$\partial_t w + \partial_x(\frac{w^2}{2}) = \gamma w \ , \tag{4.61}$$

where γ is a positive constant given in (3.7)–(3.8). We continue as in the proof of Theorem 1.2 and construct the solution w to the Burgers's equation with source

$$\partial_t w + \partial_x \left(\frac{w^2}{2}\right) = \gamma w - \kappa sgn(x) \cdot \frac{d}{dt} Q(u(t)), \tag{4.62}$$

using the same induction argument. Following similar techniques as in Subsection 4.2, Theorem 1.3 follows. □

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