CONTINUOUS DEPENDENCE OF ENTROPY SOLUTIONS TO THE EULER EQUATIONS ON THE ADIABATIC EXPONENT AND MACH NUMBER

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Abstract. We establish the L^1–estimates for continuous dependence of entropy solutions to the full Euler equations away from the vacuum on two physical parameters: The adiabatic exponent $\gamma \to 1$ that passes from the non-isentropic to isothermal Euler equations; and the Mach number $\mathrm{M} \to 0$ that passes from the compressible to incompressible Euler equations. Our analysis involves the effective approach developed in our earlier work and additional new techniques that generalize this approach to the setting of the full Euler equations.

1. Introduction

We are interested in the continuous dependence of entropy solutions to the full Euler equations on various physical parameters by scaling. In this paper, we establish the L^1–estimates for continuous dependence of entropy solutions to the full Euler equations on two physical parameters:

(i) The adiabatic exponent $\gamma \to 1$ that passes from the non-isentropic to isothermal Euler equations;

(ii) The Mach number $\mathrm{M} \to 0$ that passes from the compressible to incompressible Euler equations.

In [7], we have formulated an effective approach to compare the solutions of two different systems, which requires that one of them is endowed with the Standard Riemann Semigroup and the other has only a global entropy solution in BV obtained by the front tracking method. To be more precise, we consider the one-parameter-family of Cauchy problems:

$$\begin{cases} \partial_t W^\mu(U) + \partial_x F^\mu(U) = 0, & x \in \mathbb{R}, \\ U|_{t=0} = U_0, \end{cases} \tag{1.1}$$

where W^μ, $F^\mu : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions that depend on a parameter vector $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_i \in [0, \mu_0]$ and $U \in \mathbb{R}^n$. Assume that

(i) the particular system (1.1) for $\mu = 0$ is uniformly stable in L^1 with the Lipschitz Standard Riemann Semigroup $\mathcal{S}$ that generates the entropy solution $U(t,x) = \mathcal{S}_t U_0(x)$;

(ii) the solution U^μ to (1.1) for $\mu \neq 0$ is constructed by the front tracking algorithm.

We establish the continuous dependence in L^1 on the parameter vector μ:

$$\|U^\mu(t) - U(t)\|_{L^1} \le C\, TV\{U_0\}\, t\, \|\mu\|, \tag{1.2}$$

where $C > 0$ is a constant independent of the parameter vector μ, $TV\{U_0\}$ is the total variation of the initial data, and $\|\mu\|$ denotes the magnitude of the vector μ. Our approach utilizes both the properties of the wave front tracking algorithm and the standard error formula:

$$\|\mathcal{S}_t w(0) - w(t)\|_{L^1} \le L \int_0^t \liminf_{h \to 0+} \frac{\|\mathcal{S}_h w(\tau) - w(\tau + h)\|_{L^1}}{h} d\tau, \tag{1.3}$$

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where L is the Lipschitz constant of the semigroup $\mathcal{S}$. We have also applied the above approach to several physical systems in [7], including the L^1–estimate between the entropy solutions from the isentropic to isothermal Euler equations. The principal tools in our analysis involve the semigroup estimates as in Bressan [3]. We emphasize that the system for $\mu = 0$ is only assumed to generate a Standard Riemann Semigroup, while the system for $\mu \neq 0$ is only assumed to have a global entropy solution obtained by the front tracking method. This allows to apply our approach to certain cases where the initial data is large and in particular to treat the models of the Euler equations for which the stability of entropy solutions with large amplitude has not yet been established.

For related earlier work, Temple [26] considered the systems of form (1.1) for $\mu \neq 0$ and established the global existence by using the fact that the corresponding Glimm functional depends on the properties of the system for $\mu = 0$. Bianchini-Colombo [2] studied two systems of conservation laws with different fluxes, which both generate a Standard Riemann Semigroup and show that the semigroup is stable with respect to the C^0-norm of the Jacobian of the fluxes. Similar ideas have been used to treat the isentropic approximation to the compressible Euler system by Saint-Raymond [22]. Our approach in [7] applies even when the Standard Riemann Semigroup is constructed for only one of the two systems compared; thus it also applies to the systems with large data for which the L^1–stability of entropy solutions is still not available.

In this paper, our approach has been extended further towards two different directions: (i) the comparing systems have different number of equations, and (ii) one of the systems corresponds to the coefficients of an asymptotic expansion of the other system with respect to a physical parameter. More precisely, the purpose of this paper is to develop further the approach in [7] to establish the L^1–estimates for continuous dependence of entropy solutions to the full Euler equations on (i) the adiabatic exponent γ and (ii) the Mach number M.

(i) *From the non-isentropic (i.e. full) to isothermal Euler equations.* The main difficulty of this problem is that a system of three equations is compared with a system of two equations. It should be noted that the approach in [7] applies when dealing with two systems of the same number of equations. To overcome this difficulty, we bring into the play another system of three equations as intermediate system and apply our approach for the two systems of three equations. Having also established some additional estimates, we manage to prove the continuous dependence in L^1 for the entropy solutions of bounded variation (BV) for the non-isentropic Euler equations on the adiabatic constant $\gamma > 1$ and then to show that, as $\gamma \to 1$, the isothermal Euler equations are recovered. The key point in our analysis is to show that the contact discontinuities of the full Euler equations do not affect the limiting process.

(ii) *From the compressible to incompressible Euler equations.* The zero Mach limit $\mathrm{M} \to 0$ for smooth solutions of the compressible Euler system has been studied extensively even in the multidimensional case; see Klainerman-Majda [14], Majda [18], Metivier-Schochet [19], Schochet [23], and the references therein. Our aim here is to provide a mathematical justification of the zero Mach limit for discontinuous solutions of bounded variation for the compressible Euler equations. For this purpose, we have to restrict this problem to the one-dimensional case. The strategy is to employ our approach in this framework and show rigorously that the second order coefficients of the asymptotic expansion satisfy the linear acoustic system. The major difficulty of this problem is that the two systems have no longer form (1.1), which raises the need to modify our approach. To overcome the difficulty, we introduce a new appropriate metric and employ a general expression of error formula (1.3) that allows us now to compare between the solutions of the compressible Euler equations with small Mach number and the second order coefficients of their asymptotic expansion with respect to the Mach number.

The structure of this paper is the following. Section 2 outlines the basic properties of the full Euler equations (2.1) for compressible fluids, which are utilized in the subsequent sections. In

Section 3, we establish the L^1–estimates for continuous dependence of entropy solutions to (2.1) with large BV data away from the vacuum on the physical parameter γ. In Section 4, we establish the L^1–estimates of entropy solutions between the full Euler equations and the zero-Mach-limit equations.

2. Euler Equations for Compressible Fluids

In this section, we present some basic properties of the full Euler equations for compressible fluids, which will be used in the subsequent sections. The Euler equations generally take the following form:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(\rho\, u(E + p/\rho)) = 0, \end{cases} \tag{2.1}$$

where ρ, p, and u are the density, pressure, and velocity respectively, and

$$E = e + \frac{1}{2}u^2$$

is the total energy with e internal energy. Other important physical variables are the temperature T, entropy S, and specific volume $v = 1/\rho$. To close system (2.1), one needs the basic law of thermodynamics which translates into the differential equation:

$$T\, dS = de + p\, dv.$$

This yields the constitutive relations for this system with (v, S) as the independent variables:

$$p = -e_v(v, S) > 0, \qquad T = e_S(v, S) > 0.$$

For a polytropic gas,

$$p = R\rho T, \qquad e = c_v(\, T - T_0), \qquad c_v = \frac{R}{\gamma - 1},$$

where R and T_0 are constants. By scaling and setting

$$\varepsilon := \gamma - 1 > 0, \tag{2.2}$$

it follows that

$$p = e^{S/c_v}\rho^{\varepsilon+1}, \qquad e(\rho, S, \varepsilon) = \frac{1}{\varepsilon}\Big(\big(\frac{e^{-S/R}}{\rho}\big)^{-\varepsilon} - 1\Big). \tag{2.3}$$

Thus, as $\varepsilon \to 0$, the internal energy is given by

$$e_0(\rho, S) = \lim_{\varepsilon\to 0} e(\rho, S, \varepsilon) = \ln \rho + \frac{S}{R}.$$

An L^∞ function $U := (\rho, e, u)$ is called an entropy solution if U satisfies the Clausius-Duhem entropy inequality:

$$\partial_t(\rho\, S) + \partial_x(\rho u S) \geq 0 \tag{2.4}$$

in the sense of distributions.

System (2.1) is strictly hyperbolic when $\rho > 0$ with eigenvalues

$$\lambda_1(U, \varepsilon) = u - c, \quad \lambda_2(U, \varepsilon) = u, \quad \lambda_3(U, \varepsilon) = u + c, \tag{2.5}$$

where c is the sound speed, given by

$$c^2 = \frac{\gamma\, p}{\rho} = (1 + \varepsilon)e^{S/c_v}\rho^{\varepsilon}. \tag{2.6}$$

The Riemann invariants of (2.1) are

$$S,\ u+\tfrac{2}{\varepsilon}c, \qquad \text{1-Riemann invariants,} \tag{2.7}$$

$$u,\ p, \qquad \text{2-Riemann invariants,} \tag{2.8}$$

$$S,\ u-\tfrac{2}{\varepsilon}c, \qquad \text{3-Riemann invariants.} \tag{2.9}$$

The first and third characteristic families of (2.1) are genuinely nonlinear, while the second characteristic family is linearly degenerate. Let

$$\beta := \frac{\varepsilon+2}{\varepsilon}, \qquad \tau := \frac{\varepsilon}{2\varepsilon+2}.$$

The three family curves that join the left state (ρ_l, e_l, u_l) and right state (ρ_r, e_r, u_r) by 1–shock or rarefaction wave, a 2–contact discontinuity, and 3–shock or rarefaction wave are given in the following parametric form of parameter $y \in \mathbb{R}$:

The first family curve is

$$\rho_r = \rho_l\, f_1(y), \quad p_r = p_l\, e^{-y}, \quad u_r = u_l + c_l\, h_1(y), \tag{2.10}$$

where

$$f_1(y) \;=\; \begin{cases} e^{-\frac{y}{1+\varepsilon}}, & y \ge 0, \\ \dfrac{\beta+e^{y}}{1+\beta e^{y}}, & y \le 0, \end{cases} \tag{2.11}$$

$$h_1(y) \;=\; \begin{cases} \dfrac{2}{\varepsilon}(1-e^{-\tau\, y}), & y \ge 0, \\ \dfrac{2\sqrt{\tau}}{\varepsilon}\dfrac{1-e^{-y}}{\sqrt{1+\beta e^{-y}}}, & y \le 0. \end{cases} \tag{2.12}$$

In particular, on the first family curve,

$$e_r = e_l\frac{e^{-y}}{f_1(y)} + \frac{1}{\varepsilon}\Big(\frac{e^{-y}}{f_1(y)} - 1\Big), \tag{2.13}$$

i.e.,

$$e_r = \begin{cases} e_l e^{-\frac{\varepsilon\, y}{1+\varepsilon}} + \dfrac{1}{\varepsilon}\left(e^{-\frac{\varepsilon\, y}{1+\varepsilon}} - 1\right), & y \ge 0, \\ e_l e^{-y}\dfrac{1+\beta e^{y}}{\beta+e^{y}} + \dfrac{1}{\varepsilon}\left(\frac{e^{-y}(1+\beta e^{y})}{\beta+e^{y}} - 1\right), & y \le 0. \end{cases} \tag{2.14}$$

The second family curve is

$$\rho_r = \rho_l\, e^{\varepsilon y}, \quad p_r = p_l, \quad u_r - u_l = 0, \tag{2.15}$$

$$e_r = e_l\, e^{-\varepsilon y} + \frac{1}{\varepsilon}(e^{-\varepsilon y} - 1). \tag{2.16}$$

The third family curve is

$$\rho_r = \rho_l\, f_3(y), \quad p_r = p_l\, e^{y}, \quad u_r = u_l + c_l\, h_3(y), \tag{2.17}$$

where

$$f_3(y) \;=\; \frac{1}{f_1(y)} = \begin{cases} e^{\frac{y}{1+\varepsilon}}, & y \ge 0, \\ \dfrac{1+\beta e^{y}}{\beta+e^{y}}, & y \le 0, \end{cases} \tag{2.18}$$

$$h_3(y) \;=\; \begin{cases} \dfrac{2}{\varepsilon}(e^{\tau\, y} - 1), & y \ge 0, \\ \dfrac{2\sqrt{\tau}}{\varepsilon}\dfrac{e^{y}-1}{\sqrt{1+\beta e^{y}}}, & y \le 0. \end{cases} \tag{2.19}$$

In particular, on the third family curve,

$$e_r = e_l \frac{e^y}{f_3(y)} + \frac{1}{\varepsilon}\Big(\frac{e^y}{f_3(y)} - 1\Big), \tag{2.20}$$

i.e.,

$$e_r = \begin{cases} e_l\, e^{\frac{\varepsilon\, y}{1+\varepsilon}} + \dfrac{1}{\varepsilon}\Big(e^{\frac{\varepsilon\, y}{1+\varepsilon}} - 1\Big), & y \geq 0, \\ e_l\, e^y \dfrac{\beta + e^y}{1+\beta e^y} + \dfrac{1}{\varepsilon}\Big(\frac{e^y(\beta+e^y)}{1+\beta e^y} - 1\Big), & y \leq 0. \end{cases} \tag{2.21}$$

Note that, if $y > 0$, the first or third family curve yields a rarefaction wave connecting the left state (ρ_l, e_l, u_l) and right state (ρ_r, e_r, u_r); otherwise, if $y < 0$, either of them yields a shock wave. However, the second family curve yields a contact discontinuity for all $y \in \mathbb{R}$. The functions f_k and h_k, $k = 1, 3$, satisfy the following properties.

Lemma 2.1. *Let f_k and h_k, $k = 1, 3$, be defined as above. Then $(f_k, h_k) \in C^2$ and $(\frac{d^2 f_k}{dy^2}, \frac{d^2 h_k}{dy^2}) \in$ Lip. Moreover,*

$$f_k(0) = 1,\ h_k(0) = 0, \quad k = 1, 3,$$

and

$$\frac{df_1}{dy}(0) = -\frac{1}{1+\varepsilon}, \quad \frac{df_3}{dy}(0) = \tfrac{1}{1+\varepsilon},$$
$$\frac{dh_1}{dy}(0) = \frac{1}{1+\varepsilon}, \quad \frac{dh_3}{dy}(0) = \tfrac{1}{1+\varepsilon}.$$

More details about the wave curves can be found in Smoller [25]. The existence of entropy solutions in BV to (2.1) with Cauchy data:

$$U|_{t=0} = U_0(x) := (\rho_0, e_0, u_0)(x) \tag{2.22}$$

was established in Liu [17] and Temple [26] (also see Chen-Wagner [8]) by using Glimm's scheme [12]. This result was recently captured in Asakura [1] by the front tracking method as well, whose result can be formulated as

Lemma 2.2. *Let $N > 1$ be any positive constant and $0 < \underline{\rho} < \bar{\rho} < \infty$. Then there exists a constant $C = C(N, \underline{\rho}, \bar{\rho}) > 0$ such that, for any initial data U_0 with $\underline{\rho} \leq \rho_0(x) < \bar{\rho}$ and $TV\{U_0\} \leq N$, when*

$$\varepsilon\, TV\{U_0\} \leq C, \tag{2.23}$$

for every $\delta > 0$, the Cauchy problem (2.1) admits a δ-approximate front-tracking solution $U^{\delta,\varepsilon}$, defined for all $t \geq 0$.

Following standard arguments presented in [3], we have

Lemma 2.3. *Under the assumption of Lemma 2.2, the Cauchy problem (2.1) and (2.22) has an entropy solution $U^\varepsilon(t, x)$ defined for all $t \geq 0$, which is constructed as a limit of the wave front tracking approximations $U^{\delta,\varepsilon}$ in L^1_{loc}.*

We use the above existence result in Section 3 since we need to approximate the solution U^ε to the full Euler equations by a piecewise constant function with finitely many discontinuities. Also, the formulas of the three family curves associated with (2.1) are going to be used repeatedly in the analysis of Sections 3–4.

3. L^1–Estimates for Continuous Dependence of Entropy Solutions on the Adiabatic Exponent

In this section, we establish the L^1–estimates for continuous dependence of entropy solutions in BV to (2.1) on the physical parameter $\varepsilon = \gamma - 1 > 0$. Formally, as $\varepsilon \to 0+$, the non-isentropic Euler equations (2.1) converge to the limiting system:

$$\partial_t \rho + \partial_x(\rho\, u) = 0, \tag{3.1}$$

$$\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p_0) = 0, \tag{3.2}$$

$$\partial_t\big(\rho(\frac{1}{2}u^2 + e_0)\big) + \partial_x\big(\rho\, u(\frac{1}{2}u^2 + e_0) + p_0\, u\big) = 0, \tag{3.3}$$

with the entropy inequality (2.4), where

$$e_0 = \ln \rho + \frac{S}{R}, \qquad p_0 = \rho. \tag{3.4}$$

Observe that the above system can be decoupled into a system of two equations, the isothermal Euler equations:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \ \rho) = 0, \end{cases} \tag{3.5}$$

and an energy equation (3.3). Combining the energy equation (3.3) with the entropy inequality (2.4), we conclude the following inequality:

$$\partial_t\big(\rho(\frac{1}{2}u^2 + \ln \rho)\big) + \partial_x\big(\rho\, u(\frac{1}{2}u^2 + \ln \rho) + \rho\, u\big) \le 0 \tag{3.6}$$

in the sense of distributions, which is the corresponding entropy inequality for the isothermal Euler equations (3.5). Therefore, as $\gamma \to 1$, the non-isentropic Euler equations (2.1) reduce to the isothermal Euler equations (3.5) with the entropy condition (3.6). Thus, the aim is to compare the entropy solutions to (2.1) and (3.5) under the L^1–norm, starting out with the same initial density ρ_0 and velocity u_0. It should be noted that our approach in [7] applies when the comparing systems have the same number of equations. However, the two systems (2.1) and (3.5) that we compare here do not fulfill this condition, which raises the need to use (3.1)–(3.3) as an intermediate system. The key point here is to show that the contact discontinuity of the second family does not affect the comparison between the two systems, (2.1) and (3.5), by following our approach.

In view of the discussion in the introduction, for applying our approach to two systems of conservations laws, we require only that one of them generates a Standard Riemann Semigroup and the second system has a global entropy solution obtained by the front-tracking method. Here, the Standard Riemann Semigroup to (3.1)–(3.3) exists and the front tracking method to (2.1) converges. Indeed, the global existence of entropy solutions to (3.5) with Cauchy initial data:

$$(\rho, u)|_{t=0} = \bar{U}(x) := (\rho_0, u_0)(x) \tag{3.7}$$

was established by Nishida [20] for large initial data by Glimm's scheme. The construction of the Standard Riemann Semigroup to (3.5) was established by Colombo-Risebro in [9] when the total variation of the initial data is not necessarily small. More precisely, there exists a domain $\mathcal{D} \subseteq BV(\mathbb{R})$, a semigroup $\mathcal{S} : [0, \infty) \times \mathcal{D} \mapsto \mathcal{D}$, and a constant $L > 0$ such that $\mathcal{S}_t \bar{U}$ is the entropy solution to (3.5) with initial data $\bar{U}$ satisfying

$$\|\mathcal{S}_t \bar{U} - \mathcal{S}_s \bar{W}\|_{L^1} \le L\left(\|\bar{U} - \bar{W}\|_{L^1} + |t - s|\right).$$

Moreover, if $\bar{U}$ is piecewise constant, then, for small t, the map $(t, x) \mapsto \mathcal{S}_t \bar{U}(x)$ coincides with the solution to (3.5) obtained by piecing together the Lax solutions of the Riemann problems determined at the jumps of $\bar{U}$. Then the Standard Riemann Semigroup generated by (3.1)–(3.3) is a well defined extension of the semigroup $\mathcal{S}$ constructed by Colombo-Risebro [9] to the isothermal

Euler equations (3.5), since it corresponds to the Standard Riemann Semigroup $\mathcal{S}$ associated with (3.5) coupled with the energy equation (3.3). For notational convenience, we denote the extended Standard Riemann Semigroup associated with (3.1)–(3.3) by $\mathcal{S}$. Also, the result of Asakura [1] as stated in Lemma 2.2 implies the convergence of the front tracking method to the non-isentropic Euler equations (2.1) under the standard condition (2.23) on the initial data.

3.1. L^1 **Estimate in terms of** $\varepsilon = \gamma - 1$**.** In view of the above discussion, we now estimate the L^1-difference between the entropy solutions to the non-isentropic Euler equations (2.1) and the isothermal Euler equations (3.5) through (3.1)–(3.3) as an intermediate system.

The principal result of this section is

Theorem 3.1. *Assume that $U_0 = (\rho_0, e_0, u_0)$ satisfies the assumptions of Lemma 2.2. Let $\mathcal{S}$ be the Standard Riemann Semigroup generated by system (3.1)–(3.3) which is an extension of the Standard Riemann Semigroup generated by the isothermal Euler equations (3.5). Let $U^\varepsilon = (\rho^\varepsilon, e^\varepsilon, u^\varepsilon)$ be the entropy solution to (2.1) for $\varepsilon = \gamma - 1 > 0$ obtained by the front-tracking method. Then, for every $t > 0$,*

$$\|U^\varepsilon(t) - \mathcal{S}_t U_0\|_{L^1} = \mathcal{O}(1)\, TV\{U_0\}\, t\, \varepsilon. \tag{3.8}$$

More precisely,

$$\|(\rho^\varepsilon(t) - \rho(t), u^\varepsilon(t) - u(t))\|_{L^1} = \mathcal{O}(1)\, TV\{U_0\}\, t\, \varepsilon, \tag{3.9}$$

where $(\rho(t), u(t))$ is the solution to (3.5) and (3.7) generated by the Standard Riemann Semigroup. That is, as $\varepsilon \to 0$, for every $t > 0$, $(\rho^\varepsilon(t), u^\varepsilon(t))$ converges to the entropy solution $(\rho(t), u(t))$ of the isothermal Euler equations in L^1_{loc} with order ε, and the solution $(\rho(t), u(t))$ in the limit is the unique entropy solution within the class of viscosity solutions. Here and in the sequel, $\mathcal{O}(1)$ stands for the quantity independent of ε and t.

The proof of the theorem is carried out in the following subsections by extending our approach in [7]. Subsection 3.2 is a brief summary of the front tracking method. Subsection 3.3 is the proof of the above theorem, which is divided into several parts according to the types of discontinuities of the front tracking approximate solutions $U^{\delta,\varepsilon}$ to the non-isentropic Euler equations (2.1).

3.2. **The front tracking method.** This subsection outlines the front tracking algorithm for the use of the next subsection. For more details on the front-tracking method, we refer the reader to Bressan [3], Dafermos [11], and Holden–Risebro [13].

Let $U^{\delta,\varepsilon}$ be the front tracking approximate solutions to U^ε corresponding to the front tracking parameter $\delta > 0$. Then $U^{\delta,\varepsilon}$ is constructed as follows: First, choose a piecewise constant function $U_0^\delta(x)$ such that

$$\|U_0^\delta - U_0\|_{L^1} \le \delta. \tag{3.10}$$

Let ϱ be a constant given at the outset of the construction algorithm. At each discontinuity point of U_0^δ, a Riemann problem arises and the solution consists of shocks, contact discontinuities, and rarefaction waves. We approximate the rarefaction waves by a centered rarefaction fan containing several small jumps traveling at a speed close to the characteristic speed and with strength of each of these fronts less than δ. The piecewise constant approximate solution can be prolonged until the first set of interactions takes place. Depending on the interaction, either the Accurate Riemann Solver (ARS) or the Simplified Riemann Solver (SRS) is used to solve the Riemann problem that arises at the interaction point. ARS is basically the exact Riemann solution except that the rarefaction waves are approximated by rarefaction fans as mentioned above, while SRS introduces a non-physical wave front with constant speed $\hat{\lambda}$. ARS is employed at $t = 0$ and at every interaction between two physical waves when the product of the strengths of the incoming waves is $|\alpha\, \alpha'| \ge \varrho$, and SRS is used at every interaction involving a non-physical incoming wave-front and also at interactions where $|\alpha\, \alpha'| \le \varrho$. Hence, the algorithm involves three parameters: A fixed speed

$\hat{\lambda}$, strictly larger than all characteristic speeds; a threshold ϱ determining whether ASR or SRS is applied; and the maximum strength of rarefaction fronts which is less than the front-tracking parameter δ.

By Lemmas 2.2–2.3, $U^{\delta,\varepsilon}(t,x)$ is the δ–approximate solution for all $t \geq 0$; $U^{\delta,\varepsilon}$ is piecewise constant with a finite number of discontinuities and, as $\delta \to 0+$, $U^{\delta,\varepsilon}$ converges to the entropy solution U^{ε} of (2.1). The jumps can be of four types: shock fronts, contact discontinuities, rarefaction fronts, and non-physical fronts. The shock fronts and contact discontinuities travel with the Rankine-Hugoniot speed, while the rarefaction fronts travel with the characteristic speed of the right state and have strength less than δ. All non-physical fronts have the same speed $\hat{\lambda}$, where $\hat{\lambda}$ is a fixed constant strictly greater than all the characteristic speeds. The total strength of all non-physical fronts is uniformly small, i.e. $\sum |\alpha| < \delta$.

3.3. **Proof of Theorem 3.1.** For each $\delta > 0$, we consider the δ–front tracking approximate solution $U^{\delta,\varepsilon}$ to (2.1). Moreover, by Colombo–Risebro [9], the Standard Riemann Semigroup associated with the isothermal Euler equations (3.5) exists and generates a unique entropy solution within the class of viscosity solutions as defined in [3]. This implies that the Standard Riemann Semigroup to the limiting equations (3.1)–(3.3) exists since one first solves (3.5) and then the internal energy e_0 can be determined from the decoupled third equation (3.3). Denote by $\mathcal{S}$ the Standard Riemann Semigroup generated by (3.1)–(3.3). Thus, it suffices to show that, for each $t > 0$,

$$\|\mathcal{S}_t U_0^{\delta} - U^{\delta,\varepsilon}(t)\|_{L^1} = \mathcal{O}(1)(TV\{U_0\}\,\varepsilon + \delta)\,t \qquad \text{for all } \delta > 0, \tag{3.11}$$

since $U^{\delta,\varepsilon} \to U^{\varepsilon}$ in L^1_{loc}. By means of the error estimate (1.3), one should estimate

$$\|\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau + h)\|_{L^1} \tag{3.12}$$

for each $\tau > 0$ and $\delta > 0$. For h sufficiently small, it is equivalent to study the trajectory of the semigroup $\mathcal{S}$ to (3.1)–(3.3) with Riemann data (U_L, U_R):

$$\bar{U}(x) = \begin{cases} U_L = U^{\delta,\varepsilon}(\tau, \bar{x}-), & x < \bar{x}, \\ U_R = U^{\delta,\varepsilon}(\tau, \bar{x}+), & x > \bar{x}, \end{cases} \qquad \text{for} \quad x \in (\bar{x} - a, \bar{x} + a), \tag{3.13}$$

the states of the fronts of the approximations $U^{\delta,\varepsilon}$ at the discontinuity points $(\tau, \bar{x})$ of $U^{\delta,\varepsilon}$, but away from interaction, over the time interval $(\tau, \tau + h)$. There are four types of fronts to be investigated: shock fronts, contact discontinuities, rarefaction fronts, and non-physical fronts. To make the presentation clear and self-contained by avoiding complicate notation, we use the explicit wave curve formulas to make the estimates below.

3.3.1. **Shock front.** Let α be a shock front of the δ-approximate solution $U^{\delta,\varepsilon}$. Without loss of generality, we assume that this is a 1-shock front with left and right states:

$$U_L = (\rho_l, e_l, u_l), \qquad U_R = (\rho_r, e_r, u_r), \tag{3.14}$$

respectively. The strength α of the front is measured by

$$\alpha := \frac{\rho_r}{\rho_l}, \tag{3.15}$$

and the speed $\sigma(\varepsilon)$ is given by

$$\sigma(\varepsilon) = \frac{\rho_r\, u_r - \rho_l u_l}{\rho_r - \rho_l}. \tag{3.16}$$

Note that $|U_L - U_R| = \mathcal{O}(1)|\alpha - 1|$. The solution generated by $\mathcal{S}$ with the above Riemann data (3.14) over a time length h consists of four states:

$$U_L, \quad U_L^* = (\rho_l^*, e_l^*, u_l^*), \quad U_R^* = (\rho_r^*, e_r^*, u_r^*), \quad U_R. \tag{3.17}$$

The states are joined by three waves of strengths:

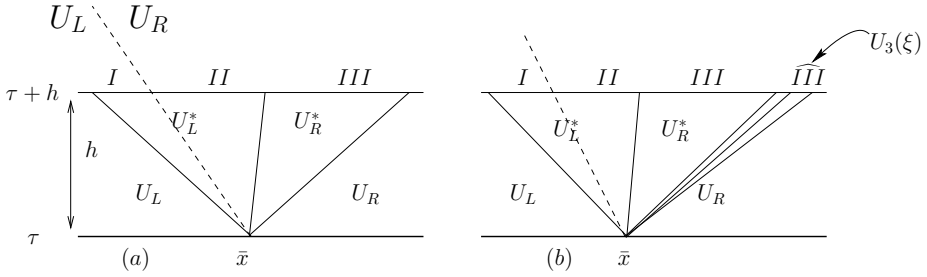

FIGURE 1. The shock front for $\gamma > 1$ and the Riemann solution for $\gamma = 1$

$$\begin{aligned}
\beta_1 &:= \frac{\rho_l^*}{\rho_l} && \text{for the 1-shock,}\\
\beta_2 &:= e_r^* - e_l^* && \text{for the 2-contact discontinuity,}\\
\beta_3 &:= \frac{\rho_r}{\rho_r^*} && \text{for the 3-shock or 3-rarefaction wave,}
\end{aligned}$$

respectively. Hence, there are two possibilities: The third wave is either a shock or a rarefaction wave, see Figure 1. Then, by the three family curves associated with (3.1)–(3.3) (cf. (2.10)–(2.21) for $\varepsilon = 0$), we have

$$\rho_l^* = \rho_r^* =: \rho^*, \qquad u_l^* = u_r^* =: u^*. \tag{3.18}$$

If we define the jump on the left state U_L and right state U_R to be

$$H_1(\alpha, \varepsilon) := (\rho_r - \rho_l,\, e_r - e_l,\, u_r - u_l)^\top, \tag{3.19}$$

then we have either

$$H_1(\alpha,\varepsilon) = \begin{pmatrix} \rho_r(\beta_1 - 1) \\ \frac{1}{2}(\beta_1 - \frac{1}{\beta_1}) \\ \frac{1-\beta_1}{\sqrt{\beta_1}} \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_l(1 - \frac{1}{\beta_3}) \\ \frac{1}{2}(\beta_3 - \frac{1}{\beta_3}) \\ \frac{\beta_3 - 1}{\sqrt{\beta_3}} \end{pmatrix}, \qquad \beta_3 \le 1, \tag{3.20}$$

or

$$H_1(\alpha,\varepsilon) = \begin{pmatrix} \rho_l(\beta_1 - 1) \\ \frac{1}{2}(\beta_1 - \frac{1}{\beta_1}) \\ \frac{1-\beta_1}{\sqrt{\beta_1}} \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_r(1 - \frac{1}{\beta_3}) \\ \ln \beta_3 \\ \ln \beta_3 \end{pmatrix}, \qquad \beta_3 \ge 1. \tag{3.21}$$

The Jacobian matrix of H_1 evaluated at $(\beta_1, \beta_2, \beta_3) = (\alpha, 0, 1)$:

$$\frac{\partial H_1}{\partial(\beta_1, \beta_2, \beta_3)} = \begin{pmatrix} \rho_l & 0 & \rho_r \\ \frac{1}{2}(1 + \frac{1}{\alpha^2}) & 1 & 1 \\ -\frac{1+\alpha}{2\alpha^{3/2}} & 0 & 1 \end{pmatrix} \tag{3.22}$$

has rank 3. It is easy to check that H is a C^2 function of $(\beta_1, \beta_2, \beta_3)$. Hence, by the Implicit Function Theorem, we have

$$\beta_1 = \beta_1(\alpha, \varepsilon), \quad \beta_2 = \beta_2(\alpha, \varepsilon), \quad \beta_3 = \beta_3(\alpha, \varepsilon). \tag{3.23}$$

Moreover,

$$\beta_1(\alpha, 0) = \alpha, \qquad \beta_2(\alpha, 0) = 0, \qquad \beta_3(\alpha, 0) = 1, \tag{3.24}$$

$$\beta_1(1, \varepsilon) = 1, \qquad \beta_2(1, \varepsilon) = 0, \qquad \beta_3(1, \varepsilon) = 1. \tag{3.25}$$

Thus,

$$\beta_1 = \alpha + \mathcal{O}(1)\varepsilon|\alpha - 1|, \quad \beta_2 = \mathcal{O}(1)\varepsilon|\alpha - 1|, \quad \beta_3 = 1 + \mathcal{O}(1)\varepsilon|\alpha - 1|. \tag{3.26}$$

Then, to estimate

$$\frac{1}{h}\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau + h)|\, dx, \tag{3.27}$$

we decompose $\{t=\tau+h\}\times(\bar{x}-a,\bar{x}+a)$ into subintervals, $I\cup II\cup III$ or $I\cup II\cup III\cup\widehat{III}$, depending on whether the third wave is a 3-shock or 3-rarefaction wave as shown in Figure 1 and compare the Riemann solution $\mathcal{S}_h\bar{U}$ with the front states U_L and U_R of $U^{\delta,\varepsilon}(\tau+h,x)$ over each subinterval for $h>0$ small. On Interval I, the integrand in (3.27) is $|U_L-U_R|$ and the length of Interval I is $h\,|\sigma(\varepsilon)-\sigma_1|$, where σ_1 is the speed of the 1-shock wave generated by $\mathcal{S}_h\bar{U}$, i.e.,

$$\sigma_1=\frac{\rho^*\,u^*-\rho_l u_l}{\rho^*-\rho_l}. \tag{3.28}$$

Since $\sigma(0)=\sigma_1$ and $\sigma(\varepsilon)\in C^2$, we obtain

$$|\sigma(\varepsilon)-\sigma_1|=\mathcal{O}(1)\,\varepsilon\qquad\text{for all }\varepsilon\in[0,\varepsilon_0].$$

Thus,

$$\frac{1}{h}\int_I|S_hU^{\delta,\varepsilon}(\tau)(x)-U^{\delta,\varepsilon}(\tau+h,x)|\,dx=\mathcal{O}(1)\,|U_L-U_R|\,\varepsilon. \tag{3.29}$$

If the third wave is a 3*-shock*, then the length of Interval $II\cup III$ is $h\,|\sigma_1(\varepsilon)-\sigma_3|=\mathcal{O}(1)\,h$, where σ_3 denotes the speed of the 3-shock. The integrand in (3.27) is $|U_L^*-U_R|$ on Interval II and $|U_R^*-U_R|$ on Interval III. Combining (3.20) with (3.26), we find

$$\begin{aligned}
&\rho^*-\rho_r=\rho_r(\tfrac{1}{\beta_3}-1)=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,\\
&e_l^*-e_r=e_l^*-e_r^*+e_r^*-e_r=\beta_2+\tfrac{1}{2}\big(\beta_3-\tfrac{1}{\beta_3}\big)=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,\\
&u^*-u_r=\tfrac{\beta_3-1}{\sqrt{\beta_3}}=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,
\end{aligned}$$

and hence

$$|U_L^*-U_R|=\mathcal{O}(1)\,|U_L-U_R|\,\varepsilon.$$

Similarly, we obtain

$$|U_R^*-U_R|=\mathcal{O}(1)\,|U_L-U_R|\,\varepsilon.$$

If the third wave is a 3*-rarefaction wave*, we denote the solution $\mathcal{S}_h\bar{U}(\tau)$ on Interval $\widehat{III}$ by $U(\xi)$ for $\xi\in[\xi_3^*,\xi_3]$. The length of Interval $II\cup III\cup\widehat{III}$ is $h\,|\sigma_1(\varepsilon)-\lambda_3(U_R,0)|=\mathcal{O}(1)\,h$, where λ_3 is given in (2.5). The integrand is $|U_L^*-U_R|$ on Interval II, which is already estimated above. Now, on Interval III, the integrand is $|U_R^*-U_R|$ and can be estimated by using the rarefaction curve. Following (3.21) and (3.26), we obtain

$$\begin{aligned}
&\rho^*-\rho_r=\rho_r(\tfrac{1}{\beta_3}-1)=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,\\
&e_r^*-e_r=-\ln\beta_3=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,\\
&u^*-u_r=-\ln\beta_3=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon.
\end{aligned}$$

Hence, in either case, the integrand is bounded by $\mathcal{O}(1)\,|U_L-U_R|\,\varepsilon$. Finally, the integrand is $|U(\xi)-U_R|$ for $\xi\in[\xi_3^*,\xi_3]$ on Interval $\widehat{III}$. Define the strength

$$\beta_3(\xi):=\frac{\rho(\xi)}{\rho^*}. \tag{3.30}$$

Therefore, we have

$$\beta_3(\xi)=\beta_3+\frac{\rho(\xi)-\rho_r}{\rho^*}=\beta_3+\mathcal{O}(1)|\xi_3^*-\xi_3|=1+\mathcal{O}(1)\,|\alpha-1|\,\varepsilon, \tag{3.31}$$

hence,

$$\begin{aligned}
&\rho(\xi)-\rho_r=\rho^*(\beta_3(\xi)-\beta_3)=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,\\
&e(\xi)-e_r=\ln\beta_3(\xi)+e_r^*-e_r=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon,\\
&u(\xi)-u_r=\ln\beta_3(\xi)+u^*-u_r=\mathcal{O}(1)\,|\alpha-1|\,\varepsilon.
\end{aligned}$$

Thus, the integrand of (3.27) over $II \cup III$ or $II \cup III \cup \widehat{III}$ is $\mathcal{O}(1)\,|U_L - U_R|\,\varepsilon$ and the length of these two Intervals is $\mathcal{O}(1)\,h$. In other words, we have

$$\frac{1}{h}\int_{II\cup III \text{ or } II\cup III\cup\widehat{III}} |S_h U^{\delta,\varepsilon}(\tau)(x) - U^{\delta,\varepsilon}(\tau+h,x)|\,dx = \mathcal{O}(1)\,|U_L - U_R|\,\varepsilon. \tag{3.32}$$

Combining (3.29) and (3.32), we arrive at

$$\frac{1}{h}\sum_{\substack{\text{shock}\\ \text{fronts}}}\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau+h)|\,dx = \mathcal{O}(1)\sum_{\substack{\text{shock}\\ \text{fronts}}} \varepsilon\,|U^{\delta,\varepsilon}(\tau,\bar{x}-) - U^{\delta,\varepsilon}(\tau,\bar{x}+)|, \tag{3.33}$$

where the summation runs over all possible shock fronts of the δ-approximate solution $U^{\delta,\varepsilon}$ at time τ. Recall that (3.33) is established at any non-interaction time τ.

3.3.2. **Contact discontinuity.** Let α be a 2-contact discontinuity of $U^{\delta,\varepsilon}$ joining the states U_L and U_R as given in (3.14). Denote the strength by

$$\alpha := e_r - e_l \tag{3.34}$$

and the speed is

$$\sigma(\varepsilon) = \lambda_2(U_L,\varepsilon) = \lambda_2(U_R,\varepsilon) = u_l = u_r. \tag{3.35}$$

Note that $|U_L - U_R| = \mathcal{O}(1)|\alpha|$. As before, the Riemann solution $\mathcal{S}_h\bar{U}$ with data (3.13) consists of four states:

$$U_L, \quad U_L^* = (\rho_l^*, e_l^*, u_l^*), \quad U_R^* = (\rho_r^*, e_r^*, u_r^*), \quad U_R, \tag{3.36}$$

with the property that $u^* := u_l^* = u_r^*$ according to the 2−family curves. The states are joined by three waves of strengths:

$$\begin{aligned}
\beta_1 &:= \frac{\rho_l^*}{\rho_l} && \text{for the 1-shock or 1-rarefaction wave,}\\
\beta_2 &:= e_r^* - e_l^* && \text{for the 2-contact discontinuity,}\\
\beta_3 &:= \frac{\rho_r}{\rho_r^*} && \text{for the 3-shock or 3-rarefaction wave,}
\end{aligned}$$

respectively.

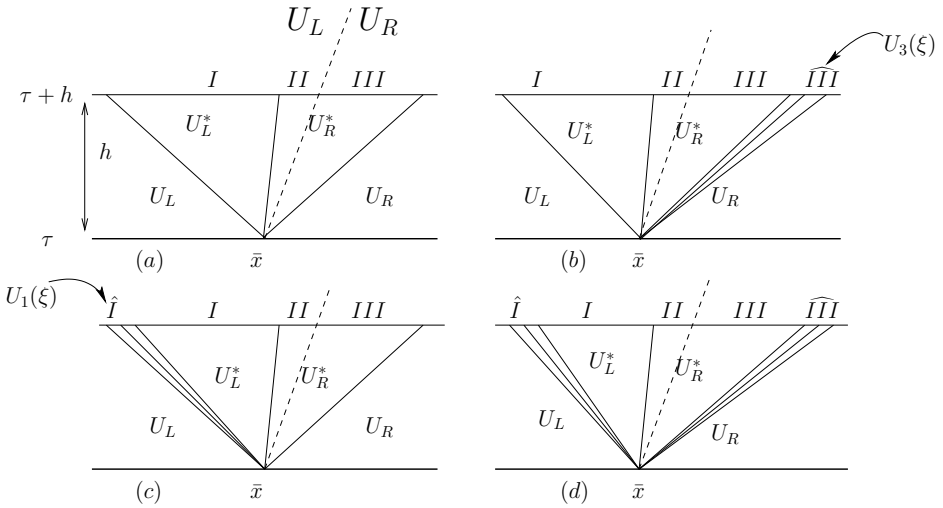

FIGURE 2. The contact discontinuity for $\gamma > 1$ and the Riemann solution for $\gamma = 1$

Hence, there are four possible Riemann solutions as shown in Figure 2. Let $U_1(\xi)$ denote the 1-rarefaction wave for $\xi \in [\xi_1, \xi_1^*] := [\lambda_1(U_L, 0), \lambda_1(U_L^*, 0)]$ on $\hat{I}$. Without loss of generality, we assume that $\bar{x} = 0$. Then $U_1(\xi) := \mathcal{S}_h \bar{U}(x)$ is a self-similar solution given by

$$\mathcal{S}_t \bar{U}(x) = \begin{cases} U_L & \text{for } \frac{x}{t} < \lambda_1(U_L), \\ R_1(s)U_L & \text{for } \frac{x}{t} = \lambda_1(R_1(s)U_L) \in [\lambda_1(U_L), \lambda_1(U_L^*)], \\ U_L^* & \text{for } \frac{x}{t} \in [\lambda_1(U_L^*), \sigma_2], \quad \sigma_2 = \lambda_2(U_L^*) = u^*, \end{cases} \tag{3.37}$$

for $t \in [0, h]$ and $x \in \hat{I} \cup I$, where $R_1(\cdot)U_L$ denotes the 1-rarefaction curve generated by the semigroup $\mathcal{S}$ associated with (3.1)–(3.3). Observe that $\hat{I} = [h\,\lambda_1(U_L, 0),\, h\,\lambda_1(U_L^*, 0)]$. Similarly, $U_3(\xi)$ is the 3-rarefaction wave for $\xi \in [\xi_3^*, \xi_3] := [\lambda_3(U_R^*, 0), \lambda_3(U_R, 0)]$ given by

$$\mathcal{S}_t \bar{U}(x) = \begin{cases} U_R^* & \text{for } \frac{x}{t} \in [\sigma_2, \lambda_3(U_R^*)], \quad \sigma_2 = \lambda_2(U_R^*) = u^*, \\ R_3(s)(U_R^*) & \text{for } \frac{x}{t} = \lambda_3(R_3(s)U_R^*) \in [\lambda_3(U_R^*), \lambda_3(U_R)], \\ U_R & \text{for } \frac{x}{t} > \lambda_3(U_R), \end{cases} \tag{3.38}$$

for $t \in [0, h]$ and $x \in \widehat{III} \cup III$.

Define again the jump on the left and right states to be

$$H_2(\alpha, \varepsilon) := (\rho_r - \rho_l,\ e_r - e_l,\ u_r - u_l)^\top. \tag{3.39}$$

Then

$$H_2(\alpha, \varepsilon) = H_{21}(\beta_1) + H_{22}(\beta_2) + H_{23}(\beta_3), \tag{3.40}$$

where

$$H_{21}(\beta_1) = \begin{cases} (\rho_l(\beta_1 - 1),\, \frac{1}{2}(\beta_1 - \frac{1}{\beta_1}),\, \frac{1 - \beta_1}{\sqrt{\beta_1}})^\top, & \beta_1 \geq 1, \\ (\rho_l(\beta_1 - 1),\, \ln \beta_1,\, -\ln \beta_1)^\top, & \beta_1 \leq 1, \end{cases} \tag{3.41}$$

$$H_{22}(\beta_2) = (0,\, \beta_2,\, 0)^\top, \tag{3.42}$$

$$H_{23}(\beta_3) = \begin{cases} (\rho_r(1 - \frac{1}{\beta_3}),\, \frac{1}{2}(\beta_3 - \frac{1}{\beta_3}),\, \frac{\beta_3 - 1}{\sqrt{\beta_3}})^\top, & \beta_3 \leq 1, \\ (\rho_r(1 - \frac{1}{\beta_3}),\, \ln \beta_3,\, \ln \beta_3)^\top, & \beta_3 \geq 1. \end{cases} \tag{3.43}$$

As before, we evaluate the Jacobian matrix of H_2 matrix at $(\beta_1, \beta_2, \beta_3) = (1, \alpha, 1)$,

$$\frac{\partial H_2}{\partial(\beta_1, \beta_2, \beta_3)} = \begin{pmatrix} \rho_l & 0 & \rho_r \\ 1 & \alpha & 1 \\ -1 & 0 & 1 \end{pmatrix}, \tag{3.44}$$

and verify that its rank is 3. Hence, the Implicit Function Theorem yields that

$$\beta_1 = \beta_1(\alpha, \varepsilon), \quad \beta_2 = \beta_2(\alpha, \varepsilon), \quad \beta_3 = \beta_3(\alpha, \varepsilon) \tag{3.45}$$

are C^2 functions of their arguments and

$$\beta_1(\alpha, 0) - 1 = \beta_2(\alpha, 0) - \alpha = \beta_3(\alpha, 0) - 1 = 0, \tag{3.46}$$

$$\beta_1(0, \varepsilon) = 1, \quad \beta_2(0, \varepsilon) = 0, \quad \beta_3(0, \varepsilon) = 1. \tag{3.47}$$

Thus,

$$\beta_1 = 1 + \mathcal{O}(1)\varepsilon|\alpha|, \quad \beta_2 = \alpha + \mathcal{O}(1)\varepsilon|\alpha|, \quad \beta_3 = 1 + \mathcal{O}(1)\varepsilon|\alpha|. \tag{3.48}$$

Again, we decompose $\{t = \tau + h\} \times (\bar{x} - a, \bar{x} + a)$ into subintervals and there are four possible cases as shown in Figure 2.

If the first wave is a 1-shock wave, then the integrand of (3.27) is $|U_L^* - U_L|$ on Interval I of length $h|\sigma_1 - \sigma_2| = \mathcal{O}(1)\, h$.

If the first wave is a 1*-rarefaction wave*, then the integrand is $|U_1(\xi) - U_L|$ on Interval $\hat{I}$ and $|U_L^* - U_L|$ on Interval I. The length of $\hat{I} \cup I$ is again $h|\lambda_1(U_L, 0) - \sigma_2| = \mathcal{O}(1)\, h$. In both cases, the difference $|U_L^* - U_L|$ can be estimated as before,

$$|U_L^* - U_L| = H_{21}(\beta_1) = \mathcal{O}(1)\,|\beta_1 - 1| = \mathcal{O}(1)\,\varepsilon|U_L - U_R|. \tag{3.49}$$

We estimate the integrand in (3.27) on Interval $\hat{I}$ as in §3.3.1, by defining

$$\beta_1(\xi) := \frac{\rho_1(\xi)}{\rho_l}, \tag{3.50}$$

to find

$$\beta_1(\xi) = \beta_1 + \frac{\rho_1(\xi) - \rho_l^*}{\rho_l} = \beta_1 + \mathcal{O}(1)|\lambda_1(U_L^*, 0) - \lambda_1(U_L, 0)| = 1 + \mathcal{O}(1)\,|\alpha|\,\varepsilon. \tag{3.51}$$

Therefore,

$$|U_1(\xi) - U_L| = |(\rho_l(\beta_1(\xi) - 1),\, -\ln \beta_1(\xi), \ln \beta_1(\xi))| = \mathcal{O}(1)\,|\alpha|\,\varepsilon. \tag{3.52}$$

Thus, the integrand is bounded by $\mathcal{O}(1)\,|\alpha|\,\varepsilon$ over I or $I \cup \hat{I}$, and the length of the interval is $\mathcal{O}(1)\, h$. Therefore, we have

$$\frac{1}{h} \int_{I \text{ or } I \cup \hat{I}} |S_h U^{\delta,\varepsilon}(\tau)(x) - U^{\delta,\varepsilon}(\tau + h, x)|\, dx = \mathcal{O}(1)\,|U_L - U_R|\,\varepsilon. \tag{3.53}$$

By symmetry, we obtain the same estimate on Intervals III and $III \cup \widehat{III}$ for the other two cases.

We carefully proceed to estimate the integral on Interval II. The speed of the β_2-wave is

$$\sigma_2 = \lambda(U_L^*, 0) = \lambda_2(U_R^*, 0) = u^*. \tag{3.54}$$

By (3.35), it follows that

$$|\sigma_2 - \sigma(\varepsilon)| = |u_l - u^*| = \mathcal{O}(1)|\beta_1 - 1| = \mathcal{O}(1)\varepsilon\,|\alpha|. \tag{3.55}$$

Hence, the length of Interval II is bounded by $\mathcal{O}(1)\, h\,\varepsilon\,|U_L - U_R|$. Now, the corresponding integrand $|U_R^* - U_L|$ on this subinterval is estimated by

$$|U_R^* - U_L| = \mathcal{O}(1)\,(|\beta_2| + |\beta_1 - 1|) = \mathcal{O}(1)\,(\varepsilon|\alpha| + |\alpha|), \tag{3.56}$$

because of (3.48). Combining (3.55) with (3.56), we arrive at

$$\int_{II} |S_h U^{\delta,\varepsilon}(\tau)(x) - U^{\delta,\varepsilon}(\tau + h, x)|\, dx = \mathcal{O}(1)\, h\,|U_L - U_R|\,\varepsilon\,(\varepsilon|\alpha| + |\alpha|). \tag{3.57}$$

To sum up, by (3.53) and (3.57), we conclude

$$\frac{1}{h} \sum_{\substack{\text{contact} \\ \text{discontinuities}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau + h)|\, dx = \mathcal{O}(1) \sum_{\substack{\text{contact} \\ \text{discontinuities}}} \varepsilon\,|U^{\delta,\varepsilon}(\tau, \bar{x}-) - U^{\delta,\varepsilon}(\tau, \bar{x}+)|, \tag{3.58}$$

where the summation is over all the fronts with contact discontinuities of the δ-approximate solution $U^{\delta,\varepsilon}$ at time τ.

3.3.3. **Rarefaction front.** This case is treated similarly to §3.3.1, the shock front. The strength of the front is measured by $\alpha := \rho_r/\rho_l$ and the speed is given by

$$\sigma(\varepsilon) = \lambda_1(U_R, \varepsilon) = u_r - c_r, \tag{3.59}$$

where $c_r^2 = \gamma\, p_r/\rho_r$. By the front tracking algorithm, $|U_R - U_L| = \mathcal{O}(1)|\alpha - 1| < \delta$. We define the map in (3.19) corresponding to this case and prove that the strengths β_1, β_2, and β_3 of the three waves determining the Riemann solution $\mathcal{S}_h\bar{U}$ can be determined by α and ε and satisfy (3.23)–(3.26). Then, we proceed in the same way as in §3.3.1 to estimate the integral over $\{t = \tau + h\} \times (\bar{x} - a, \bar{x} + a)$ and conclude

$$\frac{1}{h} \sum_{\substack{\text{rarefaction}\\ \text{fronts}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau + h)|\, dx = \mathcal{O}(1) \sum_{\substack{\text{rarefaction}\\ \text{fronts}}} \varepsilon\, |U^{\delta,\varepsilon}(\tau, \bar{x}-) - U^{\delta,\varepsilon}(\tau, \bar{x}+)|. \tag{3.60}$$

3.3.4. **Non-physical front.** By construction, the sum of strengths of these fronts is less than δ. The speed of the front is $\hat{\lambda}$, strictly greater than the characteristic speed $\lambda_3(U, \varepsilon) > 0$ for all $\varepsilon \in [0, \varepsilon_0]$, and the strength of the front is measured by the jump $\alpha := |U_L - U_R|$. Without loss of generality, we assume that the Riemann solution consists of 1, 3–shocks with strengths β_1, β_3, and a 2–contact discontinuity β_2, since we can treat possible 1, 3–rarefaction waves as before.

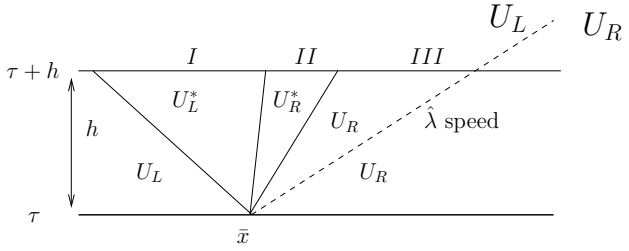

FIGURE 3. The non-physical front for $\gamma > 1$ and the Riemann solution for $\gamma = 1$

The Riemann solution $\mathcal{S}_h\bar{U}$ is shown in Figure 3. Hence, we need to estimate the integrand

$$|\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau + h)| \tag{3.61}$$

over three subintervals, I, II and III. The total length of the interval $I \cup II \cup III$ is

$$h\,|\sigma_1 - \hat{\lambda}| = \mathcal{O}(1)\, h \tag{3.62}$$

by the choice of $\hat{\lambda}$. The integrand attains three possible values, one over each interval; $|U_L^* - U_L|$ on I, $|U_R^* - U_L|$ on II, and $|U_R - U_L|$ on III. We claim that each value is bounded by the strength α of the non-physical front.

Indeed, recall that $\mathcal{S}$ denotes the Standard Riemann Semigroup associated with limiting system (3.1)–(3.3), which decouples to the isothermal Euler equations (3.5) and a linear first order equation (3.3). One of the fundamental properties of system (3.5) is that the total variation of the solution is non-increasing in time even if the initial data is large. This was first proved by Nishida [20] to establish the global entropy solutions to (3.5). This implies that

$$\beta_1 = 1 + \mathcal{O}(1)\,|U_L - U_R|, \qquad \beta_3 = 1 + \mathcal{O}(1)\,|U_L - U_R|, \tag{3.63}$$

$$|\rho_l - \rho^*| + |\rho^* - \rho_r| + |u_l - u^*| + |u^* - u_r| = \mathcal{O}(1)\,|U_L - U_R|. \tag{3.64}$$

To complete the claim, it suffices to check the quantities $|e_l^* - e_l|$ and $|e_r^* - e_l|$. Indeed,

$$|e_l^* - e_l| = \frac{1}{2}\big|\beta_1 - \frac{1}{\beta_1}\big| = \mathcal{O}(1)|U_L - U_R|, \tag{3.65}$$

$$|e_r^* - e_l| = |e_r^* - e_r| + |e_r - e_l| = \frac{1}{2}\big|\beta_3 - \frac{1}{\beta_3}\big| + |e_r - e_l| = \mathcal{O}(1)|U_L - U_R|, \tag{3.66}$$

by the bounds in (3.64), which leads to the claim.

Thus, by (3.62) and the bounds on the integrand over each subinterval, we conclude

$$\frac{1}{h} \sum_{\substack{\text{non-physical}\\ \text{fronts}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau+h)|\, dx = \mathcal{O}(1) \sum_{\substack{\text{non-physical}\\ \text{fronts}}} |U^{\delta,\varepsilon}(\tau,\bar{x}-) - U^{\delta,\varepsilon}(\tau,\bar{x}+)| < \delta, \tag{3.67}$$

since the total strength of non-physical fronts is less than the front tracking parameter $\delta > 0$.

By (3.33), (3.58), (3.60), and (3.67), we conclude

$$\begin{aligned} \|\mathcal{S}_t U^{\delta,\varepsilon}(0) - U^{\delta,\varepsilon}(t)\|_{L^1} &\le L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau+h)|\, dx d\tau \\ &= L\ \mathcal{O}(1)\Big((\varepsilon+\delta)\int_0^t TV U^{\delta,\varepsilon}(\tau)\, d\tau + \delta\, t\Big) \\ &= \mathcal{O}(1)\ (\varepsilon\, TV\{U_0\} + \delta)t. \end{aligned} \tag{3.68}$$

Thus, as $\delta \to 0$, we obtain the L^1–estimate (3.8) for continuous dependence on $\varepsilon = \gamma - 1$. Since system (3.1)–(3.3) consists of system (3.5) and the energy equation (3.3), estimate (3.9) follows. The statement on the convergence of the density and velocity of the non-isentropic Euler equations to the corresponding variables of the isothermal Euler equations as $\varepsilon \to 0$ is an immediate consequence of (3.9). Also, the entropy solution $(\rho,\, u)(t,x)$ in the limit is unique within the class of *viscosity solutions*; this is an immediate consequence of the existence of the Standard Riemann Semigroup $\mathcal{S}$ (cf. Bressan [3]). The proof is complete.

4. L^1–Estimates for Continuous Dependence of Entropy Solutions on the Mach number

The non-isentropic Euler equations for gas dynamics with small Mach number take the following form:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + \tfrac{1}{\mathrm{M}^2} p) = 0, \\ &\partial_t(\rho E) + \partial_x((\rho E + \tfrac{1}{\mathrm{M}^2} p)u) = 0, \end{aligned} \tag{4.1}$$

where ρ is the density of the gas, u the velocity, p the pressure and

$$E = \frac{1}{\mathrm{M}^2}\frac{p}{(\gamma-1)\rho} + \frac{u^2}{2} \tag{4.2}$$

the energy. The parameter $\mathrm{M} > 0$ denotes the Mach number. For convenience, in this case, we use the independent variables $U = (\rho, p, u)$ instead of (ρ, e, u).

System (4.1) is derived by the non-dimensional scaling of system (2.1) with Cauchy initial data $U_0 = (\rho_0, p_0, u_0)(x)$:

$$(t,x) \longrightarrow (|u_m|t, x), \qquad (\rho, p, u) \longrightarrow (\frac{\rho}{\rho_m}, \frac{p}{\gamma p_m}, \frac{u}{|u_m|}),$$

where $\rho_m = \max(\rho_0(x))$, $p_m = \max(p_0(x))$, and $|u_m| = \max|u_0(x)|$. Then the Mach number in (4.1) is

$$\mathrm{M} = \frac{|u_m|}{c_m},$$

where $c_m = \frac{\gamma p_m}{\rho_m}$. In many physical situations such as air at room temperature, M is rather small. So it is important to understand the limiting behavior of entropy solutions to (4.1) as $\mathrm{M} \to 0$.

Consider a sequence of Cauchy data for (4.1) of the following form:

$$\begin{aligned} \rho|_{t=0} &= \rho_0 + \rho_2^{(0)}(x)\,\mathrm{M}^2, \\ p|_{t=0} &= p_0 + p_2^{(0)}(x)\,\mathrm{M}^2, \\ u|_{t=0} &= u_1^{(0)}(x)\,\mathrm{M}, \end{aligned} \tag{4.3}$$

with $(\rho_2^{(0)}, p_2^{(0)}, u_1^{(0)}) \in BV(\mathbb{R};\mathbb{R}^3)$, where p_0 and ρ_0 are positive constants. Denote the solution to (4.1) and (4.3) by $(\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}})$. Formal analysis (cf. Majda [18]) shows that the solution $(\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}})$ has an asymptotic expansion:

$$\begin{aligned} \rho^{\mathrm{M}}(t,x) &= \rho_0 + \rho_2^{\mathrm{M}}(t,x)\,\mathrm{M}^2 + O(1)\mathrm{M}^3, \\ p^{\mathrm{M}}(t,x) &= p_0 + p_2^{\mathrm{M}}(t,x)\,\mathrm{M}^2 + O(1)\mathrm{M}^3, \\ u^{\mathrm{M}}(t,x) &= u_1^{\mathrm{M}}(t,x)\,\mathrm{M} + O(1)\mathrm{M}^2, \end{aligned} \tag{4.4}$$

where $(\rho_2^{\mathrm{M}}, p_2^{\mathrm{M}}, u_1^{\mathrm{M}})$ is the solution to the following linear acoustic system:

$$\begin{aligned} &\partial_t \rho_2 + \frac{\rho_0}{\mathrm{M}} \partial_x u_1 = 0, \\ &\partial_t p_2 + \frac{\gamma p_0}{\mathrm{M}} \partial_x u_1 = 0, \\ &\partial_t u_1 + \frac{1}{\mathrm{M}\rho_0} \partial_x p_2 = 0, \end{aligned} \tag{4.5}$$

with the initial data

$$(\rho_2, p_2, u_1)\big|_{t=0} = (\rho_2^{(0)}, p_2^{(0)}, u_1^{(0)})(x). \tag{4.6}$$

4.1. L^1**–Estimates in terms of the Mach Number** M**.** We now give the rigorous justification of the asymptotic expansion (4.4) in the BV setting.

Theorem 4.1. *Let* $(\rho_2^{(0)},\, p_2^{(0)},\, u_1^{(0)}) \in BV(\mathbb{R};\mathbb{R}^3)$. *Then there exists a constant* $M_0 > 0$ *such that, when* $M \in (0, M_0)$, *the Cauchy problem* (4.1) *and* (4.3) *has a global solution* $(\rho^M, p^M, u^M)(t,x) \in BV(\mathbb{R}^2_+;\mathbb{R}^3)$. *Moreover, for every* $t \geq 0$,

$$\|\rho^M(t) - \rho_0 - \rho_2^M(t)\,M^2\|_{L^1} = \mathcal{O}(1)\,TV\{(\rho_2^{(0)},\, p_2^{(0)},\, u_1^{(0)})\}\,t\,M^3, \tag{4.7}$$

$$\|p^M(t) - p_0 - p_2^M(t)\,M^2\|_{L^1} = \mathcal{O}(1)\,TV\{(\rho_2^{(0)},\, p_2^{(0)},\, u_1^{(0)})\}\,t\,M^3, \tag{4.8}$$

$$\|u^M(t) - u_1^M(t)\,M\|_{L^1} = \mathcal{O}(1)\,TV\{(\rho_2^{(0)},\, p_2^{(0)},\, u_1^{(0)})\}\,t\,M^2, \tag{4.9}$$

where (ρ_2^M, p_2^M, u_1^M) *is the unique weak solution to* (4.5)–(4.6). *Here and in the sequel of this section,* $\mathcal{O}(1)$ *stands for the quantity independent of* M *and* t.

The problem on the zero Mach limits of the solutions to the compressible Euler system has been studied extensively by many authors, for instance, see Majda [18] and the references therein. In [18], it is shown that the solution has the incompressible part as the coefficient of the first order in M. By asymptotic analysis, it indicates that the coefficient of the second order in M satisfies the linear acoustic systems. The mathematical justification of this expansion in the context of *smooth solutions* to the full Euler equations even in the multidimensional case was provided in Klainerman-Majda [14].

In §4.2–4.3, we provide the mathematical justification of the expansion in the context of *discontinuous solutions* only in BV. To achieve this, we further develop our approach in [7] and then we

apply the new approach on systems (4.1) and (4.5). However, we can only treat the one-dimensional case since the BV theory of conservation laws is only available for one-space dimension problems.

Our strategy is the following: We first define an appropriate metric so that the error formula still holds under this metric space and employ the Standard Riemann Semigroup $\mathcal{S}$ associated with (4.1) for small initial data which is available as indicated in Lemma 4.4. Then we also approximate the unique weak solution to (4.5) by a sequence of piecewise continuous functions $(\rho_2^{\mathrm{M},k}, p_2^{\mathrm{M},k}, u_1^{\mathrm{M},k})$ with finite number of discontinuities that converges to the solution to (4.5) as $k \to \infty$. It should be noted that, because of the linearity of system (4.5), we do not need to employ the front tracking algorithm for this construction. Finally, we apply the semigroup $\mathcal{S}$ on the approximations $(\rho_2^{\mathrm{M},k}, p_2^{\mathrm{M},k}, u_1^{\mathrm{M},k})$ in the error formula. Hence, for each $k < \infty$, we study carefully the Riemann solution to (4.1) at the "initial time" τ where the fronts of $(\rho_2^{\mathrm{M},k}, p_2^{\mathrm{M},k}, u_1^{\mathrm{M},k})$ are away from interaction times. We also get the L^1–estimate on the difference between the trajectory of semigroup and the approximate solutions that have only contact discontinuities. The procedure is similar to the one presented in Section 3.

4.2. **Weak solutions to the linear acoustic system** (4.5)**.** We first solve the Cauchy problem (4.5)–(4.6) and then state the corresponding Rankine-Hugoniot conditions for weak solutions.

Let

$$v_1 = p_2 + \frac{\gamma p_0}{c_0} u_1, \qquad v_2 = p_2 - \frac{\gamma p_0}{c_0} u_1, \tag{4.10}$$

where $c_0 = \sqrt{\gamma\, p_0/\rho_0}$. Then, the last two equations of (4.5) can be rewritten as

$$\partial_t v_1 + \frac{c_0}{\mathrm{M}} \partial_x v_1 = 0,$$
$$\partial_t v_2 - \frac{c_0}{\mathrm{M}} \partial_x v_2 = 0,$$

while the first two equations of (4.5) imply

$$\partial_t \rho_2 - \frac{\rho_0}{\gamma p_0} \partial_t p_2 = 0.$$

Then it is direct to conclude that there exists the following unique weak solution $(\rho_2^{\mathrm{M}}, p_2^{\mathrm{M}}, u_1^{\mathrm{M}})$ to the Cauchy problem (4.5)–(4.6):

$$\rho_2^{\mathrm{M}} = \frac{\rho_0}{\gamma p_0} p_2^{\mathrm{M}}(x,t) + \rho_2^{(0)}(x) - \frac{\rho_0}{\gamma p_0} p_2^{(0)}(x),$$
$$p_2^{\mathrm{M}} = \frac{1}{2}\Big(p_2^{(0)}(x - \frac{c_0}{\mathrm{M}} t) + p_2^{(0)}(x + \frac{c_0}{\mathrm{M}} t)\Big) + \frac{\gamma p_0}{2c_0}\Big(u_1^{(0)}(x - \frac{c_0}{\mathrm{M}} t) - u_1^{(0)}(x + \frac{c_0}{\mathrm{M}} t)\Big),$$
$$u_1^{\mathrm{M}} = \frac{1}{2}\Big(u_1^{(0)}(x - \frac{c_0}{\mathrm{M}} t) + u_1^{(0)}(x + \frac{c_0}{\mathrm{M}} t)\Big) + \frac{1}{2c_0\rho_0}\Big(p_2^{(0)}(x - \frac{c_0}{\mathrm{M}} t) - p_2^{(0)}(x + \frac{c_0}{\mathrm{M}} t)\Big).$$

Moreover, since the linear acoustic system is strictly hyperbolic with eigenvalues

$$\lambda_1 = -\frac{c_0}{\mathrm{M}}, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{c_0}{\mathrm{M}}, \tag{4.11}$$

the weak solution $(\rho_2^{\mathrm{M}},\, p_2^{\mathrm{M}},\, u_2^{\mathrm{M}})$ to (4.5) consists of three types of discontinuous curves:

$$x + \frac{c_0}{\mathrm{M}} t = const., \quad x - \frac{c_0}{\mathrm{M}} t = const., \quad x = const.$$

In the next lemma, we derive the Rankine-Hugoniot conditions across the jump of discontinuity. The proof is omitted since it follows from straightforward calculations.

Lemma 4.2. *For any two constant states $(\rho_{2L}, p_{2L}, u_{1L})$ and $(\rho_{2R}, p_{1R}, u_{1R})$, with $\rho_{2L} > 0$, $\rho_{2R} > 0$, $p_{2L} > 0$, and $p_{2R} > 0$, denote*

$$\rho_2^*(x) = \begin{cases} \rho_{2L}, & x < \bar{x}, \\ \rho_{2R}, & x > \bar{x}, \end{cases} \qquad p_2^*(x) = \begin{cases} p_{2L}, & x < \bar{x}, \\ p_{2R}, & x > \bar{x}, \end{cases} \qquad u_1^*(x) = \begin{cases} u_{1L}, & x < \bar{x}, \\ u_{1R}, & x > \bar{x}. \end{cases}$$

Let $[\rho_2^*] = \rho_{2R} - \rho_{2L}$, $[p_2^*] = p_{2R} - p_{2L}$, and $[u_1^*] = u_{1R} - u_{1L}$ be the corresponding jumps. Then,

(i) $c_0^2[\rho_2^*] = [p_2^*] = -c_0\,\rho_0[u_1^*]$ *if and only if the function*

$$W_-(x,t) = \begin{cases} (\rho_{2L}, p_{2L}, u_{1L}), & x + \frac{c_0}{M}t < \bar{x}, \\ (\rho_{2R}, p_{2R}, u_{1R}), & x + \frac{c_0}{M}t > \bar{x} \end{cases}$$

is the solution to (4.5) with initial data $W_-(x,0) = (\rho_2^*(x), p_2^*(x), u_1^*(x))$;

(ii) $c_0^2[\rho_2^*] = [p_2^*] = c_0\,\rho_0[u_1^*]$ *if and only if the function*

$$W_+(x,t) = \begin{cases} (\rho_{2L}, p_{2L}, u_{1L}), & x - \frac{c_0}{M}t < \bar{x}, \\ (\rho_{2R}, p_{2R}, u_{1R}), & x - \frac{c_0}{M}t > \bar{x} \end{cases}$$

is the solution to (4.5) with initial data $W_+(x,0) = (\rho_2^*(x), p_2^*(x), u_1^*(x))$;

(iii) $[p_2^*] = [u_1^*] = 0$ *if and only if the function*

$$W_m(x,t) = \begin{cases} (\rho_{2L}, p_{1L}, u_{1L}), & x < \bar{x}, \\ (\rho_{2R}, p_{1R}, u_{1R}), & x > \bar{x} \end{cases}$$

is the solution to (4.5) with initial data $W_m(x,0) = (\rho_2^*(x), p_1^*(x), u_1^*(x))$.

4.3. **Riemann problem for (4.1).** As discussed in Section 2, the scaled Euler system (4.1) is strictly hyperbolic with three distinct eigenvalues (cf. (2.5)):

$$\lambda_1^{\mathrm{M}}(\rho,p,u) = u - \frac{c}{\mathrm{M}}, \qquad \lambda_2^{\mathrm{M}}(\rho,p,u) = u, \qquad \lambda_3^{\mathrm{M}}(\rho,p,u) = u + \frac{c}{\mathrm{M}}, \tag{4.12}$$

when $\rho > 0$, $p > 0$, and $c = \sqrt{\gamma\, p/\rho}$. For any state $U = (\rho, p, u)$ with $\rho > 0$ and $p > 0$, consider the three family curves associated with system (4.1):

$$\begin{aligned} \Phi_1^{\mathrm{M}}(U;y) &= \big(\rho f_1(y), pe^{-y}, u + \frac{c}{\mathrm{M}}h_1(y)\big), \\ \Phi_2^{\mathrm{M}}(U;y) &= \big(\rho e^{y}, p, u\big), \\ \Phi_3^{\mathrm{M}}(U;y) &= \big(\rho f_3(y), pe^{y}, u + \frac{c}{\mathrm{M}}h_3(y)\big), \end{aligned}$$

where f_k and h_k, $k = 1, 3$, are given in (2.11)–(2.12) and (2.18)–(2.19). Set

$$T_k(\rho_0, p_0, u_0) := \{(\rho,p,u) \,:\, (\rho,p,u) = \Phi_k^{\mathrm{M}}(\rho_0, p_0, u_0; y)\}, \quad k = 1, 3.$$

Then $T_k(\rho_0, p_0, u_0)$ is the $k-$wave curve emanating from (ρ_0, p_0, u_0), where the parameter $y > 0$ implies that (ρ, p, u) and (ρ_0, p_0, u_0) can be connected by a rarefaction wave with (ρ_0, p_0, u_0) as the left state, while $y < 0$ implies that (ρ, p, u) and (ρ_0, p_0, u_0) can be connected by a shock wave with (ρ_0, p_0, u_0) as the left state. For $k = 2$, the two states are connected by a contact discontinuity for any $y \in \mathbb{R}$.

Now, we consider the Riemann problem for (4.1) with initial data:

$$(\rho,p,u)\big|_{t=0} = \begin{cases} (\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}), & x < \bar{x}, \\ (\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}}), & x > \bar{x}, \end{cases} \tag{4.13}$$

where

$$\begin{aligned} &\rho_L^{\mathrm{M}} = \rho_0 + \rho_{2L}\,\mathrm{M}^2, \qquad p_L^{\mathrm{M}} = p_0 + p_{2L}\,\mathrm{M}^2, \qquad u_L^{\mathrm{M}} = u_{1L}\,\mathrm{M}, \\ &\rho_R^{\mathrm{M}} = \rho_0 + \rho_{2R}\,\mathrm{M}^2, \qquad p_R^{\mathrm{M}} = p_0 + p_{2R}\,\mathrm{M}^2, \qquad u_R^{\mathrm{M}} = u_{1R}\,\mathrm{M}, \end{aligned} \tag{4.14}$$

for $(\rho_{2L},\, p_{2L},\, u_{1L})$ and $(\rho_{2R},\, p_{2R},\, u_{1R})$ independent of M. As in Section 3, to determine the Riemann solution, it suffices to find a solution (y_1, y_2, y_3) to the vector equation:

$$\Phi_3^{\mathrm{M}}(\Phi_2^{\mathrm{M}}(\Phi_1^{\mathrm{M}}(\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}; y_1); y_2); y_3) = (\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}}). \tag{4.15}$$

This is equivalent to solving the equation

$$H(y_1, y_2, y_3) = G, \tag{4.16}$$

where

$$H(y_1, y_2, y_3) = \begin{pmatrix} f_1(y_1) f_3(y_3) e^{y_2} \\ y_3 - y_1 \\ h_1(y_1) + \frac{e^{-(y_1+y_2)/2}}{\sqrt{f_1(y_1)}} h_3(y_3) \end{pmatrix}, \quad G = \left(\frac{\rho_R^{\mathrm{M}}}{\rho_L^{\mathrm{M}}},\, \ln \frac{p_R^{\mathrm{M}}}{p_L^{\mathrm{M}}},\, \mathrm{M} \frac{u_R^{\mathrm{M}} - u_L^{\mathrm{M}}}{c_L^{\mathrm{M}}} \right)^{\top}, \tag{4.17}$$

and $c_L^{\mathrm{M}} := \sqrt{\gamma p_L^{\mathrm{M}} / \rho_L^{\mathrm{M}}}$. Indeed, the next lemma states that there is a solution (y_1, y_2, y_3) to (4.15) for sufficiently small M.

Lemma 4.3. *For sufficiently small M, the equation*

$$H(y_1, y_2, y_3) = G \tag{4.18}$$

has a unique solution (y_1, y_2, y_3) near $y_1 = y_2 = y_3 = M = 0$. Moreover, the solution satisfies the following estimates:

(i) *If $c_0^2[\rho_2^*] = [p_2^*] = -c_0\rho_0[u_1^*]$, then*

$$y_1 = \frac{\gamma M^2}{c_0}[u_1^*] + \mathcal{O}(1)[u_1^*]\, M^4, \qquad y_2 = y_3 = \mathcal{O}(1)[u_1^*]\, M^4;$$

(ii) *If $c_0^2[\rho_2^*] = [p_2^*] = c_0\rho_0[u_1^*]$, then*

$$y_1 = y_2 = \mathcal{O}(1)[u_1^*]\, M^4, \qquad y_3 = \frac{\gamma M^2}{c_0}[u_1^*] + \mathcal{O}(1)[u_1^*]\, M^4;$$

(iii) *If $[p_2^*] = [u_1^*] = 0$, then*

$$y_2 = \frac{M^2}{\rho_0}[\rho_2^*] + \mathcal{O}(1)[\rho_2^*]\, M^4, \qquad y_1 = y_3 = \mathcal{O}(1)[\rho_2^*]\, M^4,$$

where ρ_2^*, p_2^*, and u_1^* are defined in Lemma 4.2.

Proof. Let H and G be given in (4.17). By direct calculation, we verify that H and G are C^2 functions of their arguments for small M and

$$\frac{\partial H}{\partial (y_1, y_2, y_3)}\Big|_{y_1=y_2=y_3=0} = \begin{pmatrix} -1/\gamma & 1 & 1/\gamma \\ -1 & 0 & 1 \\ 1/\gamma & 0 & 1/\gamma \end{pmatrix}. \tag{4.19}$$

We also rewrite G in (4.17) by means of (4.14) as

$$G = \begin{pmatrix} 1 + \frac{\rho_{2R} - \rho_{2L}}{\rho_L^{\mathrm{M}}}\, \mathrm{M}^2 \\ \ln\left(1 + \frac{p_{2R} - p_{2L}}{p_L^{\mathrm{M}}}\, \mathrm{M}^2\right) \\ \frac{u_{1R} - u_{1L}}{c_L^{\mathrm{M}}}\, \mathrm{M}^2 \end{pmatrix}. \tag{4.20}$$

Hence, the Implicit Function Theorem yields a unique C^2 solution (y_1, y_2, y_3) to (4.18) for small M. To prove statement (i), we define

$$G_*(\delta,\eta) := \begin{pmatrix} 1 - \frac{\rho_0\,\delta/c_0}{\rho_0+\eta\rho_{2L}} \\ \ln\Big(1 - \frac{c_0\,\rho_0\,\delta}{p_0+\eta p_{2L}}\Big) \\ \frac{\delta}{c_L(\eta)} \end{pmatrix} \tag{4.21}$$

and

$$c_L(\eta) := \sqrt{\frac{\gamma(p_0+\eta p_{2L})}{\rho_0+\eta\rho_{2L}}}.$$

Note that, if $\delta = \mathrm{M}^2[u_1^*]$, then $\mathrm{M}^2[p_2^*] = -c_0\,\rho_0\,\delta$ and $\mathrm{M}^2[\rho_2^*] = -\rho_0\,\delta/c_0$; hence, $G_*(\mathrm{M}^2[u_1^*], \mathrm{M}^2) = G$.

Now, we consider the following equation:

$$H(\alpha_1,\alpha_2,\alpha_3) = G_*(\delta,\eta). \tag{4.22}$$

Applying again the Implicit Function Theorem gives a unique solution $(\alpha_1,\alpha_2,\alpha_3)$ to (4.22) for small δ and η. Moreover, $(\alpha_1,\alpha_2,\alpha_3)$ is a C^2 function of (δ,η) satisfying

$$\alpha_1(0,\eta) = \alpha_2(0,\eta) = \alpha_3(0,\eta) = 0. \tag{4.23}$$

Observe that, when $\delta = \mathrm{M}^2[u_1^*]$ and $\eta = \mathrm{M}^2$,

$$(\alpha_1,\alpha_2,\alpha_3) = (y_1,y_2,y_3). \tag{4.24}$$

Differentiating (4.22) with respect to δ and evaluating at $(\delta,\eta) = (0,0)$ yields

$$\frac{\partial H}{\partial(\alpha_1,\alpha_2,\alpha_3)}\partial_\delta \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \frac{1}{c_0}\begin{pmatrix} -1 \\ -\gamma \\ 1 \end{pmatrix}.$$

Hence,

$$\partial_\delta\alpha_2|_{\delta=\eta=0} = \partial_\delta\alpha_3\big|_{\delta=\eta=0} = 0, \qquad \partial_\delta\alpha_1\big|_{\delta=\eta=0} = \frac{\gamma}{c_0}. \tag{4.25}$$

Therefore, by (4.23), we deduce

$$\begin{aligned} \alpha_1(\delta,\eta) &= \big(\alpha_1(\delta,\eta) - \alpha_1(\delta,0) - \alpha_1(0,\eta) + \alpha_1(0,0)\big) + \alpha_1(\delta,0) \\ &= O(1)|\delta||\eta| + \alpha_1(\delta,0) \\ &= \frac{\gamma\delta}{c_0} + O(1)|\delta||\eta| + O(1)|\delta|^2, \end{aligned}$$

$$\begin{aligned} \alpha_2(\delta,\eta) &= \big(\alpha_2(\delta,\eta) - \alpha_2(\delta,0) - \alpha_2(0,\eta) + \alpha_2(0,0)\big) + \alpha_2(\delta,0) \\ &= O(1)|\delta||\eta| + \alpha_2(\delta,0) \\ &= O(1)|\delta||\eta| + O(1)|\delta|^2, \end{aligned}$$

and similarly for α_3. The above estimates imply statement (i) due to (4.24).

The proof of statements (ii) and (iii) can be carried out in the same way. The proof is complete. ■

4.4. L^1–**stability estimates for** (4.1)**.** We now present the stability property of (4.1) and the error formula corresponding to (1.3) under an appropriate metric, as well as the corresponding estimates when applied to the contact discontinuities that the solution to the linear acoustic system (4.5) admits.

Lemma 4.4. *Suppose that* $(\rho_0^k,\, p_0^k,\, u_0^k) = (\rho_0+\rho_2^{(0),k}{}_M{}^2, p_0+p_2^{(0),k}{}_M{}^2, u_1^{(0),k}{}_M)$ *with constants* $\rho_0, p_0 > 0$, *satisfy*

$$TV(\rho_2^{(0),k}, p_2^{(0),k}, u_1^{(0),k}) < \infty, \qquad k = 1, 2.$$

Then there exists $M_0 > 0$ *such that, if* $0 < M < M_0$, *there exist global solutions*

$$(\rho^{k,M}, p^{k,M}, u^{k,M})(t, x)$$

to (4.1) *with the initial data*

$$(\rho, p, u)\Big|_{t=0} = (\rho_0^k, p_0^k, u_0^k)(x), \tag{4.26}$$

for $k = 1, 2$, *respectively. Moreover,*

$$\begin{aligned} &M||u^{1,M}(t) - u^{2,M}(t)||_{L^1} + ||\rho^{1,M}(t) - \rho^{2,M}(t)||_{L^1} + ||p^{1,M}(t) - p^{2,M}(t)||_{L^1} \\ &= \mathcal{O}(1)\Big(M||u^{1,M}(0) - u^{2,M}(0)||_{L^1} + ||\rho^{1,M}(0) - \rho^{2,M}(0)||_{L^1} + ||p^{1,M}(0) - p^{2,M}(0)||_{L^1}\Big). \end{aligned} \tag{4.27}$$

Proof. Following the rescaling: $(\tilde{t}, \tilde{u}) = (t/M, Mu)$ and $(\tilde{\rho}, \tilde{p}) = (\rho, p)$, system (4.1) is equivalent to

$$\begin{cases} \partial_{\tilde{t}}\widetilde{\rho} + \partial_x(\widetilde{\rho}\,\widetilde{u}) = 0, \\ \partial_{\tilde{t}}(\widetilde{\rho}\widetilde{u}) + \partial_x(\widetilde{\rho}\,\widetilde{u}^2 + \widetilde{p}) = 0, \\ \partial_{\tilde{t}}(\widetilde{\rho}\widetilde{E}) + \partial_x((\widetilde{\rho}\,\widetilde{E} + \widetilde{p})\widetilde{u}) = 0, \end{cases} \tag{4.28}$$

for

$$\widetilde{E} = \frac{\widetilde{p}}{(\gamma-1)\widetilde{\rho}} + \frac{\widetilde{u}^2}{2},$$

with initial data

$$(\widetilde{\rho}, \widetilde{p}, \widetilde{u})\Big|_{\tilde{t}=0} = (\rho_0^k, p_0^k, Mu_0^k)(x). \tag{4.29}$$

By the standard results of Glimm [12] (also see [3, 11, 24, 25]), there exists a small positive constant M_0 such that, when $M < M_0$,

$$TV(\rho_0^k) + TV(p_0^k) < M_0^2, \quad TV(u_0^k) < M_0, \tag{4.30}$$

and hence there exists a global entropy solution $(\widetilde{\rho}^{k,M}, \widetilde{p}^{k,M}, \widetilde{u}^{k,M})$ to (4.28)–(4.29) for $k = 1, 2$.

Now, the solution to (4.1) and (4.26) is given by

$$(\rho^{k,M}, p^{k,M}, u^{k,M})(t, x) = (\widetilde{\rho}^{k,M}, \widetilde{p}^{k,M}, \frac{1}{M}\widetilde{u}^{k,M})(\frac{t}{M}, x), \qquad k = 1, 2. \tag{4.31}$$

Thus, the existence of global solutions to (4.1) and (4.26) is guaranteed under condition (4.30) for $k = 1, 2$.

Moreover, the L^1–stability of solutions to systems of conservation laws with small data has been established (cf. [3, 5, 11]). Thus, for the compressible Euler equations (4.28)–(4.29),

$$\begin{aligned} &\|\widetilde{u}^{1,M}(\tau) - \widetilde{u}^{2,M}(\tau)\|_{L^1} + \|\widetilde{\rho}^{1,M}(\tau) - \widetilde{\rho}^{2,M}(\tau)\|_{L^1} + \|\widetilde{p}^{1,M}(\tau) - \widetilde{p}^{2,M}(\tau)\|_{L^1} \\ &\quad = \mathcal{O}(1)\Big(||\widetilde{u}^{1,M}(0) - \widetilde{u}^{2,M}(0)||_{L^1} + ||\widetilde{\rho}^{1,M}(0) - \widetilde{\rho}^{2,M}(0)||_{L^1} + ||\widetilde{p}^{1,M}(0) - \widetilde{p}^{2,M}(0)||_{L^1}\Big) \end{aligned} \tag{4.32}$$

for any $\tau \geq 0$, which implies the stability estimate (4.27). The proof is complete. ∎

It should be noted that, in Lemma 4.4, we do not require any smallness assumptions on the total variation of $(\rho_2^{(0)}, p_2^{(0)}, u_1^{(0)})$, since the initial data in (4.3) has small total variation when $M < M_0$.

As indicated in [3], the above lemma implies that there exists $\mathrm{M}_0 > 0$ such that (4.1)–(4.3) has a global entropy solution $(\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}}) \in BV$ for $\mathrm{M} \in (0, \mathrm{M}_0)$ with a Lipschitz Standard Riemann Semigroup S_t^{M} in L^1; and the domain of the semigroup is

$$D = \{V = (\rho, p, u) \,:\, (\rho - \rho_0, p - p_0, u) \in L^1,\, TV(\rho) + TV(p) < \mathrm{M}^2,\, TV(u) < \mathrm{M}\}.$$

We define the metric

$$d_{\mathrm{M}}(V, \widetilde{V}) = ||\rho - \tilde{\rho}||_{L^1} + ||p - \tilde{p}||_{L^1} + \mathrm{M}||u - \tilde{u}||_{L^1} \tag{4.33}$$

for any $V = (\rho, p, u), \widetilde{V} = (\tilde{\rho}, \tilde{p}, \tilde{u}) \in D$. Then, from [3], we quote the following fact.

Lemma 4.5. *There exists a constant L independent of M such that*

$$d_M(S_t^M U(0), U(t)) \le L \int_0^t \liminf_{h \to 0+} \frac{d_M(S_h^M U(\tau), U(\tau + h))}{h} d\tau \tag{4.34}$$

for any $U \in D$.

We remark that, in fact, the constant L depends only on $\mathcal{O}(1)$ in (4.32).

Let W_-, W_m, and W_+ be the functions defined in Lemma 4.2. Define

$$U_{\mp}^{\mathrm{M}}(t,x) = \begin{cases} (\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}), & x < \bar{x} \pm \frac{c_0}{\mathrm{M}} t, \\ (\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}}), & x > \bar{x} \pm \frac{c_0}{\mathrm{M}} t, \end{cases}$$

$$U_m^{\mathrm{M}}(t,x) = \begin{cases} (\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}), & x < \bar{x}, \\ (\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}}), & x > \bar{x}, \end{cases}$$

where $(\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}})$ and $(\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}})$ are given in (4.13)–(4.14). The following lemma establishes the main estimates needed to prove Theorem 4.1.

Lemma 4.6. *Let $U_{\pm}^M(t,x)$ and $U_m^M(t,x)$ be defined as above, and ρ_2^*, p_2^*, and u_1^* be as those in Lemma 4.2. Let $\mathcal{S}_t^M$ be the Standard Riemann Semigroup associated with (4.1). Then there exists a positive constant L_1, independent of M, $[\rho_2^*]$, $[p_2^*]$, and $[u_1^*]$, such that the following estimates hold:*

(i) *If $c_0^2[\rho_2^*] = [p_2^*] = -c_0\,\rho_0\,[u_1^*]$, then*

$$d_M(S_t^M U_-^M(0), U_-^M(t)) \le L_1 [u_1^*]_{\mathrm{M}}{}^3 t; \tag{4.35}$$

(ii) *If $c_0^2[\rho_2^*] = [p_2^*] = c_0\,\rho_0\,[u_1^*]$, then*

$$d_M(S_t^M U_+^M(0), U_+^M(t)) \le L_1 [u_1^*]_{\mathrm{M}}{}^3 t; \tag{4.36}$$

(iii) *If $[p_2^*] = [u_1^*] = 0$, then*

$$d_M(S_t^M U_m^M(0), U_m^M(t)) \le L_1 [\rho_2^*]_{\mathrm{M}}{}^3 t. \tag{4.37}$$

Proof. We first consider the case that $c_0^2[\rho_2^*] = [p_1^*] = -c_0\,\rho_0[u_1^*]$. Then, by Lemma 4.2,

$$W_-(t,x) = (\rho_2^{\mathrm{M}}, p_2^{\mathrm{M}}, u_1^{\mathrm{M}})(t,x) = \begin{cases} (\rho_{2R}, p_{2R}, u_{2R}) & \text{for } x > \bar{x} + \frac{c_0}{M} t, \\ (\rho_{2L}, p_{2L}, u_{2L}) & \text{for } x < \bar{x} + \frac{c_0}{M} t \end{cases} \tag{4.38}$$

is the unique solution to (4.5)–(4.6). Then

$$U_-^{\mathrm{M}}(t,x) = (\rho_w^{\mathrm{M}}(t,x), p_w^{\mathrm{M}}(t,x), u_w^{\mathrm{M}}(t,x)), \tag{4.39}$$

where $\rho_w^{\mathrm{M}}(t,x) = \rho_0 + \rho_2^{\mathrm{M}}(t,x)\,\mathrm{M}^2$, $p_w^{\mathrm{M}}(t,x) = p_0 + p_2^{\mathrm{M}}(t,x)\,\mathrm{M}^2$, and $u_w^{\mathrm{M}}(t,x) = u_1^{\mathrm{M}}(t,x)\,\mathrm{M}$ are the first two terms in the expansion (4.4).

By Lemma 4.3, for sufficiently small M, (4.15) has a unique solution (y_1, y_2, y_3) that satisfies

$$y_1 = \frac{\gamma_{\mathrm{M}}{}^2}{c_0} [u_1^*] + \mathcal{O}(1)[u_1^*]\,\mathrm{M}^4, \qquad y_2 = y_3 = O(1)[u_1^*]\,\mathrm{M}^4. \tag{4.40}$$

Equivalently, the solution $S_t^{\mathrm{M}}U_-^{\mathrm{M}}(0) = (\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}})(t)$ to (4.1) with Riemann data:

$$U_-^{\mathrm{M}}\Big|_{t=0} = \begin{cases} (\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}}) & \text{for } x > \bar{x}, \\ (\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}) & \text{for } x < \bar{x} \end{cases} \tag{4.41}$$

exists and consists of four states

$$U_L^{\mathrm{M}}, \quad U_{m,1}^{\mathrm{M}}, \quad U_{m,2}^{\mathrm{M}}, \quad U_R^{\mathrm{M}}, \tag{4.42}$$

joined by three waves with strengths y_1, y_2, and y_3 respectively, where $U_{m,j}^{\mathrm{M}} = (\rho_{m,j}^{\mathrm{M}}, p_{m,j}^{\mathrm{M}}, u_{m,j}^{\mathrm{M}})$, $j = 1, 2$, and $U_{m,1}^{\mathrm{M}} = \Phi_1(\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}, y_1)$, while $U_{m,2}^{\mathrm{M}} = \Phi_2(U_{m,1}^{\mathrm{M}}, y_2)$. By the family curves of (4.1) as presented in §4.3, we deduce that the pressure and velocity remain constant along the second family, i.e. $p_m^{\mathrm{M}} := p_{m,1}^{\mathrm{M}} = p_{m,2}^{\mathrm{M}}$ and $u_m^{\mathrm{M}} := u_{m,1}^{\mathrm{M}} = u_{m,2}^{\mathrm{M}}$, while the density satisfies

$$\rho_{m,2}^{\mathrm{M}} = \rho_{m,1}^{\mathrm{M}} e^{y_2} = \rho_{m,1}^{\mathrm{M}}\big(1 + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^4\big), \tag{4.43}$$

by using (4.40). Moreover, by (4.40), we have

$$\rho_{m,1}^{\mathrm{M}} = \rho_L^{\mathrm{M}} f_1(y_1) = \rho_L^{\mathrm{M}}\big(1 + \frac{\mathrm{M}^2}{c_0}[u_1^*] + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^4\big), \tag{4.44}$$

$$p_m^{\mathrm{M}} = p_L^{\mathrm{M}} e^{-y_1} = p_L^{\mathrm{M}}\big(1 - \frac{\mathrm{M}^2}{c_0}[u_1^*] + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^4\big), \tag{4.45}$$

$$\begin{aligned} u_m^{\mathrm{M}} &= u_L^{\mathrm{M}} + \frac{1}{\mathrm{M}} c_L^{\mathrm{M}} h_1(y_1) = u_L^{\mathrm{M}} + \gamma[u_1^*]_{\mathrm{M}} + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^2 \\ &= u_{1L\,\mathrm{M}} + \gamma[u_1^*]_{\mathrm{M}} + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^2, \end{aligned} \tag{4.46}$$

and

$$\rho_R^{\mathrm{M}} = \rho_{m,2}^{\mathrm{M}} f_3(y_3) = \rho_{m,2}^{\mathrm{M}}(1 + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^4), \tag{4.47}$$

$$p_R^{\mathrm{M}} = p_m^{\mathrm{M}} e^{y_3} = p_m^{\mathrm{M}}(1 + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^4), \tag{4.48}$$

$$u_R^{\mathrm{M}} = u_m^{\mathrm{M}} + \frac{1}{\mathrm{M}} c_{m,2}^{\mathrm{M}} h_3(y_3) = u_m^{\mathrm{M}} + \mathcal{O}(1)[u_1^*]_{\mathrm{M}}^3. \tag{4.49}$$

Furthermore, we estimate

$$c_L^{\mathrm{M}} - c_0 = \Big(\frac{\gamma(p_0 + p_{1L}\mathrm{M}^2)}{\rho_0 + \rho_{1L}\mathrm{M}^2}\Big)^{1/2} - \Big(\frac{\gamma p_0}{\rho_0}\Big)^{1/2} = \mathcal{O}(1)\mathrm{M}^2, \tag{4.50}$$

$$\lambda_1^{\mathrm{M}}(U_L^{\mathrm{M}}) - (-\frac{c_0}{\mathrm{M}}) = u_L^{\mathrm{M}} - \frac{c_L^{\mathrm{M}}}{\mathrm{M}} - (-\frac{c_0}{\mathrm{M}}) = u_{1L\mathrm{M}} - \frac{1}{\mathrm{M}}(c_L^{\mathrm{M}} - c_0) = \mathcal{O}(1)_{\mathrm{M}}. \tag{4.51}$$

Let

$$\begin{aligned} q_1^- &= \min\{\lambda_1^{\mathrm{M}}(U_L^{\mathrm{M}}),\, \lambda_1^{\mathrm{M}}(U_{m,1}^{\mathrm{M}}),\, -\frac{c_0}{\mathrm{M}}\}, \\ q_2^- &= \max\{\lambda_1^{\mathrm{M}}(U_L^{\mathrm{M}}),\, \lambda_1^{\mathrm{M}}(U_{m,1}^{\mathrm{M}}),\, -\frac{c_0}{\mathrm{M}}\}, \\ q^+ &= \max\{\lambda_3^{\mathrm{M}}(U_{m,2}^{\mathrm{M}}),\, \lambda_3^{\mathrm{M}}(U_R^{\mathrm{M}})\}. \end{aligned}$$

Combining (4.43)–(4.51), we arrive at

$$q_2^- - q_1^- = \mathcal{O}(1)_{\mathrm{M}}, \qquad q^+ - q_1^- = \mathcal{O}(1)\frac{1}{\mathrm{M}}. \tag{4.52}$$

For every $t \geq 0$, it remains to estimate the difference of $\mathcal{S}_t^{\mathrm{M}}U_-^{\mathrm{M}}(0) = (\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}})(t)$ and $U_-^{\mathrm{M}}(t) = (\rho_w^{\mathrm{M}}, p_w^{\mathrm{M}}, u_w^{\mathrm{M}})(t)$ in L^1. In Figure 4, we sketch the two functions around the discontinuity point $x = \bar{x}$. As it is shown, there are two possible Riemann solutions: The third wave is either a shock $y_3 < 0$ or a rarefaction wave $y_3 > 0$. Then, at time t, $U_-^{\mathrm{M}}(t)$ consists of the front (dashed line) joining

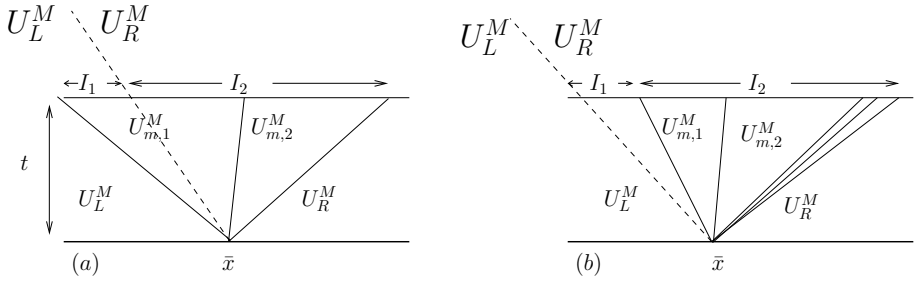

FIGURE 4. The contact discontinuity for (4.5) and the Riemann solution to (4.1)

$U_L^{\rm M}$ and $U_R^{\rm M}$ with speed $\sigma = -c_0/\mathrm{M}$. On the other hand, $\mathcal{S}_t^{\rm M} U_-^{\rm M}(0)$ attains different values along Interval $I_1 \cup I_2$. It is easy to check that

$$\text{length of } I_1 \le (q_2^- - q_1^-)t = \mathcal{O}(1)t\,\mathrm{M}, \tag{4.53}$$

and

$$\text{length of } I_2 \le (q^+ - q_1^-)t = \mathcal{O}(1)\frac{t}{\mathrm{M}}. \tag{4.54}$$

Over Interval I_1, the value of $|\rho^{\rm M}(t) - \rho_w^{\rm M}(t)|$ is either $|\rho_L^{\rm M} - \rho_R^{\rm M}|$ or $|\rho_{m,1}^{\rm M} - \rho_L^{\rm M}|$. Over Interval I_2, the possible values are $|\rho_{m,1}^{\rm M} - \rho_R^{\rm M}|$ or $|\rho_{m,2}^{\rm M} - \rho_R^{\rm M}|$. Thus,

$$\begin{aligned} ||\rho^{\rm M}(t) - \rho_w^{\rm M}(t)||_{L^1} &\le \int_{I_1} \left(|\rho_L^{\rm M} - \rho_{m,1}^{\rm M}| + |\rho_L^{\rm M} - \rho_R^{\rm M}|\right) dx + \int_{I_2} \sum_{j=1,2} |\rho_{m,j}^{\rm M} - \rho_R^{\rm M}| dx \\ &= \mathcal{O}(1)[u_1^*]t\,\mathrm{M}^3, \end{aligned} \tag{4.55}$$

by using (4.43)–(4.44), (4.47), and (4.53)–(4.54). Similarly,

$$\|p^{\rm M}(t) - p_w^{\rm M}(t)\|_{L^1} = \mathcal{O}(1)[u_1^*]t\,\mathrm{M}^3, \tag{4.56}$$

$$\|u^{\rm M}(t) - u_w^{\rm M}(t)\|_{L^1} = \mathcal{O}(1)[u_1^*]t\,\mathrm{M}^2. \tag{4.57}$$

Estimates (4.55)–(4.57) imply statement (i). Statements (ii)–(iii) can be proved in a similar way. Here, we omit their proof since, in Section 3, we have presented the corresponding cases for the non-isentropic Euler equations in detail. The proof is complete. ■

4.5. **Proof of Theorem 4.1.** We approximate the vector function $(\rho_2^{(0)}, p_2^{(0)}, u_1^{(0)})(x)$ by piecewise constant functions $(\rho_{2,k}^{(0)}, p_{2,k}^{(0)}, u_{1,k}^{(0)})(x)$ such that

$$\|(\rho_{2,k}^{(0)} - \rho_2^{(0)}, p_{2,k}^{(0)} - p_2^{(0)}, u_{1,k}^{(0)} - u_1^{(0)})\|_{L^1} \le 1/k,$$

and we assume that $(\rho_{2,k}^{(0)}, p_{2,k}^{(0)}, u_{1,k}^{(0)})$ have a finite number of discontinuity points, but no common discontinuities. Then, following §4.2, for each k, there exists a unique global weak solution $W^{{\rm M},k}(t,x) = (\rho_{2,k}^{\rm M}, p_{2,k}^{\rm M}, u_{1,k}^{\rm M})(t,x)$ to (4.5) with initial data

$$W^{{\rm M},k}\big|_{t=0} = (\rho_{2,k}^{(0)}, p_{2,k}^{(0)}, u_{1,k}^{(0)})(x).$$

Consider the set of times $\Lambda_{{\rm M},k} = \{t \ : \ \text{there are two fronts in } W^{{\rm M},k} \text{ crossing at time } t\}$. Then, $\Lambda_{{\rm M},k} \cap [0,T]$ is a finite set for any $T < \infty$ and k.

Define the approximate functions $U^{{\rm M},k}$ associated with the non-isentropic Euler equations to be

$$\begin{aligned} U^{{\rm M},k}(t,x) &= (\rho_{w,k}^{\rm M}(t,x), p_{w,k}^{\rm M}(t,x),\, u_{w,k}^{\rm M}(t,x)) \\ &= (\rho_0 + \mathrm{M}^2 \rho_{2,k}^{\rm M}(t,x), p_0 + \mathrm{M}^2 p_{2,k}^{\rm M}(t,x), \mathrm{M} u_{1,k}^{\rm M}(t,x)). \end{aligned} \tag{4.58}$$

We apply the error formula on these approximations $U^{\mathrm{M},k}$,

$$d_{\mathrm{M}}(S_t^{\mathrm{M}}U^{\mathrm{M},k}(0), U^{\mathrm{M},k}(t)) \le L\int_0^t \liminf_{h\to 0+} \frac{d_{\mathrm{M}}(S_h^{\mathrm{M}}U^{\mathrm{M},k}(\tau), U^{\mathrm{M},k}(\tau+h))}{h} d\tau. \tag{4.59}$$

We claim that, for any $t_0 \in (0,\infty)\backslash\Lambda_{\mathrm{M},k}$,

$$d_{\mathrm{M}}\left(S_h^{\mathrm{M}}U^{\mathrm{M},k}(t_0),\, U^{\mathrm{M},k}(t_0+h)\right) = \mathcal{O}(1)\, h\, {\mathrm{M}}^3 \tag{4.60}$$

for sufficiently small h. Indeed, since $t_0 \in (0,\infty)\backslash\Lambda_{\mathrm{M},k}$, then, for sufficiently small $h>0$, there is no front in $W^{\mathrm{M},k}$ that crosses within the strip in the t–x plane bounded by $t=t_0$ and $t=t_0+h$. To estimate (4.60), it suffices to solve the Riemann problem for the Euler equations (4.1) that is generated by $\mathcal{S}^{\mathrm{M}}$ over a time step h having, as Riemann data, the fronts

$$\bar{U}(x) = \begin{cases} U^{\mathrm{M},k}(t_0, \bar{x}-) = (\rho_L^{\mathrm{M}}, p_L^{\mathrm{M}}, u_L^{\mathrm{M}}) & \text{for } x\in(x-a,\bar{x}), \\ U^{\mathrm{M},k}(t_0, \bar{x}+) = (\rho_R^{\mathrm{M}}, p_R^{\mathrm{M}}, u_R^{\mathrm{M}}) & \text{for } x\in(\bar{x}, x+a) \end{cases} \tag{4.61}$$

of the approximations $U^{\mathrm{M},k}$ at time $t_0 \in (0,\infty)\backslash\Lambda_{\mathrm{M},k}$ at each discontinuity point $(\bar{x},t_0)$ of $U^{\mathrm{M},k}$. Then (4.60) is an immediate consequence of the results in Lemma 4.6 for a time step h, since we have

$$d_{\mathrm{M}}\left(S_h^{\mathrm{M}}U^{\mathrm{M},k}(t_0),\, U^{\mathrm{M},k}(t_0+h)\right) \le L_1\, h\, TV\{W^{\mathrm{M},k}(t_0)\}\, {\mathrm{M}}^3, \tag{4.62}$$

which implies the claim. Notice that the factor $TV\{W^{\mathrm{M},k}(t_0)\}$ in (4.62) arises since the summation is over all the fronts of $W^{\mathrm{M},k}(t_0)$. Moreover, $TV\{W^{\mathrm{M},k}(t_0)\} = \mathcal{O}(1)\, TV\{(\rho_2^{(0)}, p_2^{(0)}, u_1^{(0)})\}$ for almost all $t_0>0$. As $k\to\infty$, combining (4.59) and (4.62) with (4.33), we conclude (4.7)–(4.9). Theorem 4.1 follows. □

Remark 1. *We remark that Theorem* 4.1 *can also be proved by following a different strategy but similar approach to the one presented here. Instead, we can choose the front tracking approximate solutions* $U^{\delta,M}$ *to the compressible Euler equations* (4.1) *with Mach number* $\mathrm{M}>0$ *and the Lipschitz Standard Riemann Semigroup* $\mathcal{S}_t^M$ *associated with to the linear acoustic limit* (4.5)*. Then, using the techniques presented in this section, we can translate the solution obtained by this semigroup and compare it with the approximations* $U^{\delta,M}$*. Although the Riemann problem to the linear acoustic limit* (4.5) *is easier to solve, the approach involves the front tracking method for* (4.1) *which is more difficult since the non-physical fronts of* $U^{\delta,M}$ *to* (4.1) *then have to be handled; while, in the proof above, we avoid this since the front tracking method is not needed.*

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