THE DYNAMICAL BOREL-CANTELLI LEMMA FOR INTERVAL MAPS

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Abstract. The dynamical Borel-Cantelli lemma for some interval maps is considered. For expanding maps whose derivative has bounded variation, any sequence of intervals satisfies the dynamical Borel-Cantelli lemma. If a map has an indifferent fixed point, then the dynamical Borel-Cantelli lemma does not hold even in the case that the map has a finite absolutely continuous invariant measure and summable decay of correlations.

1. Introduction. Let \((X, \mu)\) be a probability space and \(B_n\) be a sequence of subsets of \(X\). There are two kinds of classical Borel-Cantelli lemmas: The first lemma is that if \(\sum \mu(B_n) < \infty\), then for almost every \(x\) there are finitely many \(n\)'s such that \(x \in B_n\). The second one is that if \(\sum \mu(B_i) = \infty\) and \(B_n\)'s are independent, then for almost every \(x\) there are infinitely many \(n\)'s such that \(x \in B_n\). Let \(T\) be a \(\mu\) preserving transformation on \(X\) and \(A_n\) be a sequence of subsets in \(X\) with \(\sum \mu(A_n) = \infty\). If we put \(B_n = T^{-n}A_n\) and \(B_n\)'s are independent, then by the second Borel-Cantelli lemma \(T^n x \in A_n\) for infinitely many \(n\)'s.

Assume that \(A_n\) is a sequence of subsets of \(X\) with \(\sum \mu(A_n) = \infty\). A sequence of subsets \(A_n \subset X\) is called a Borel-Cantelli (BC)\(^1\) sequence if for \(\mu\)-almost every \(x \in X\) there are infinitely many \(n\)'s such that \(T^n x \in A_n\).

Let \(S_N(x)\) be the number of positive integers \(1 \leq n \leq N\) such that \(T^n(x) \in A_n\), i.e.,

\[ S_N(x) = \sum_{n=1}^{N} 1_{A_n} \circ T^n(x) = \sum_{n=1}^{N} 1_{T^{-n}A_n}(x), \]

where \(1_{A_n}(x)\) is the indicator of the set \(A_n\). We set

\[ E_N = \mu(S_N) = \sum_{n=1}^{N} \mu(A_n). \]

We call a sequence of subsets \(A_n \subset X\) a strongly Borel-Cantelli (SBC) sequence if

\[ \lim_{N \to \infty} S_N(x)/E_N = 1 \]

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\(^1\)The notions of BC and SBC are from [3].
for $\mu$-almost every $x$. Obviously, an SBC sequence is BC.

Note that a measure preserving transformation $T$ is ergodic if and only if every constant sequence $A_n = A$, $\mu(A) > 0$, is BC. See [3] for the relation with weakly mixing property.

The first proof of the dynamical Borel-Cantelli lemma has been given by Philipp. Let $X = [0, 1)$ be the unit interval. Suppose that $T(x) = rx \pmod{1}$, $r > 1$ or $T(x) = \{1/x\}$ and that $\mu$ is the unique $T$-invariant absolutely continuous measure on $X$. Let $A_n$ be a sequence of intervals in $X$ with $\sum \mu(A_n) = \infty$. Philipp [12] showed that then

$$S_N(x) = \mu(S_N) + O(\mu(S_N)^{1/2} \log^{3/2+\epsilon} \mu(S_N)), \quad \epsilon > 0$$

for almost all $x \in X$.

Chernov and Kleinbock later showed that every sequence of various shapes of balls are SBC for an Anosov diffeomorphism [3]. See [4] for partially hyperbolic systems.

We extend the definition of SBC sequences for sequences of nonnegative functions: for a sequence of nonnegative functions $f_n$ on $X$, put

$$S_N(x) = \sum_{n=1}^{N} f_n \circ T^n(x)$$

and

$$E_N = \mu(S_N) = \sum_{n=1}^{N} \mu(f_n).$$

We call the sequence of nonnegative functions $f_n$ on $X$ a strongly Borel-Cantelli (SBC) sequence if

$$\lim_{N \to \infty} S_N(x)/E_N = 1,$$

for almost every $x$. A sequence of subsets $A_n$ is SBC if and only if the indicator function $1_{A_n}(x)$ is SBC. The following lemma is an important tool for the proof.

**Lemma 1.1** ([14]). Let $(\Omega, \mu)$ be a measure space, let $f_k(\omega) \ (k = 1, 2, \ldots)$ be a sequence of nonnegative $\mu$-measurable functions, and let $\bar{f}_k, \varphi_k$ be sequences of real numbers such that

$$0 \leq \bar{f}_k \leq \varphi_k \leq M \quad (k = 1, 2, \ldots).$$

Suppose that

$$\int_{\Omega} \left( \sum_{m < k \leq n} f_k(\omega) - \sum_{m < k \leq n} \bar{f}_k \right)^2 d\mu \leq C \sum_{m < k \leq n} \varphi_k$$

for arbitrary integers $m$, $n \ (m < n)$. Then

$$\sum_{1 \leq k \leq n} f_k(\omega) = \sum_{1 \leq k \leq n} \bar{f}_k + O(\Phi^{1/2}(n) \ln^{3/2+\epsilon} \Phi(n))$$

for almost all $\omega \in \Omega$, where $\epsilon > 0$ is arbitrary and $\Phi(n) = \sum_{1 \leq k \leq n} \varphi_k$.

In this paper expanding maps on the interval are considered. For expanding maps whose derivative has bounded variation, every sequence of intervals satisfies the dynamical Borel-Cantelli lemma. When a map is not uniformly expanding, the dynamical Borel-Cantelli lemma does not hold for sequences of intervals in general, even in the case that the map has a finite absolutely continuous invariant measure and summable decay of correlations.
In Section 2 we consider the class of piecewise monotone transformations on the interval. In Section 3 and 4 we apply the theorem in previous sections to non-uniformly expanding maps.

2. Uniformly expanding maps. In this section we consider piecewise expanding maps on the unit interval \( X = [0, 1] \). Let \( I = \{(a_i, b_i)\}_{i=1}^{\infty} \) be a countable family of closed intervals with disjoint interiors such that \( \bigcup I = X \). Let \( U = \bigcup (a_i, b_i) \) and \( S = X \setminus U \). We assume that \( T \) is differentiable on each \((a_i, b_i)\), \( T \) is called piecewise expanding if \(|T'(x)| \geq \alpha > 1\). The Perron-Frobenius operator is defined as

\[
Pf(x) = \sum_{y \in T^{-1}\{x\}} g(y)f(y),
\]

where

\[
g(x) = \begin{cases} 
    \frac{1}{|T'(x)|} & \text{if } x \in U, \\
    0 & \text{if } x \in S.
\end{cases}
\]

We assume that \( T \) has a unique absolutely continuous invariant measure \( d\mu = hdx \) and the dynamical system \((T, \mu)\) is weakly mixing.

Let \( BV(X) \) be the set of bounded variation functions over \( X \) endowed with the norm \( \|f\|_{BV} = \int f + \|f\|_1 \), where \( \int f \) denotes the variation of a function \( f \). Rychlik[13] (see also [6]) showed that if \( T \) is piecewise expanding and \( g \) is of bounded variation, then there exist a positive constant \( C_0 \) and a constant \( 0 < r < 1 \) such that for any \( f \in BV(X) \)

\[
\left\| P^n(f) - \left( \int f \right) h \right\|_{BV} \leq C_0 r^n \|f\|_{BV}.
\]

Thus there are \( r < 1 \) and \( C > 0 \) such that for all \( n \geq 1 \) and all \( f \in L^1(\mu) \) and \( \psi \in BV(X) \),

\[
\begin{align*}
\left| \int f \circ T^n \psi d\mu - \int f d\mu \int \psi d\mu \right| & \leq \|f\|_1 \|P^n(\psi h) - h(\int \psi h dx)\|_{BV} \\
& \leq C r^n \|f\|_1 \|\psi\|_{BV},
\end{align*}
\]

where \( C = C_0 (\int h + \|h\|_{\infty}) \). See also [1] for a comprehensive reference.

**Theorem 2.1.** Let \( T \) be a piecewise expanding map with bounded variation \( g = 1/|T'| \). Assume that \( T \) has a uniquely absolutely continuous invariant measure \( d\mu = hdx \) and \( h \) is bounded away from 0. If \( f_n \) is a sequence of nonnegative functions with \( \sum \mu(f_n) = \infty \) and \( \|f_n\|_{BV} < M \) for some \( M \), then for almost every \( x \)

\[
\lim_{N \to -\infty} \frac{\sum_{n=1}^{N} f_n \circ T^n(x)}{\sum_{n=1}^{N} \mu(f_n)} = 1,
\]

i.e., every sequence of uniformly bounded variation functions \( f_n \) is SBC.

**Proof.** Let \( f_n \) be a sequence of nonnegative functions on \( X \) with \( \|f_n\|_{BV} < M \). Then for \( i < j \),

\[
\int (f_j \circ T^i(x)) \cdot (f_i \circ T^i(x)) d\mu = \int (f_j \circ T^{j-i}(x)) \cdot f_i(x) d\mu.
\]

By (1) for a positive constant \( D \) we have

\[
\left| \int (f_j \circ T^{j-i}(x)) \cdot f_i(x) d\mu - \mu(f_j)\mu(f_i) \right| \leq CMr^{j-i} \|f_j\|_1 \leq DMr^{j-i} \mu(f_j).
\]
Hence
\[ \int \left( \sum_{m<k \leq n} f_k(x) - \sum_{m<k \leq n} \mu(f_k) \right)^2 \, d\mu = \sum_{m<i,j \leq n} \left( \int f_i(x)f_j(x) \, d\mu - \mu(f_i)\mu(f_j) \right) < (M + 2DM \frac{r}{1-r}) \sum_{m<k \leq n} \mu(f_k). \]

Put \( \varphi_k = \tilde{f}_k = \mu(f_k) \). Then by Lemma 1.1, we complete the proof.

Therefore, every sequence of intervals is SBC with respect to the invariant measure \( \mu \).

3. Borel-Cantelli lemma for induced transformations. Let \( \mu \) be a probability measure on \( X \) and \( T : X \to X \) be a \( \mu \)-preserving transformation. For a measurable subset \( E \subset X \) with \( \mu(E) > 0 \) and a point \( x \in E \) which returns to \( E \) under iteration by \( T \), we define \( R_E \) to be the first return time
\[ R_E(x) = \min \{ j \geq 1 : T^j(x) \in E \}. \]

Kac’s lemma[9] states that
\[ \int E R_E(x) \, d\mu(x) \leq 1, \]
where the equality holds if \( T \) is ergodic. Let \( T_E \) be the induced transformation of \( T \) on \( E \), which is defined by
\[ T_E(x) = T^{R_E(x)}(x). \]

Then \( T_E : E \to E \) preserves \( \mu \). Note that if \( T \) is ergodic, then \( T_E \) is also ergodic.

**Theorem 3.1.** Suppose that \( T \) is ergodic. Let \( f_n \) be a sequence of nonnegative functions such that \( f_1 \geq f_2 \geq \cdots \geq 0 \), \( \sum_n \mu(f_n) = \infty \) and \( \text{supp}(f_n) \subset E \). If every subsequence \( f_{n_k} \) with \( \sum \mu(f_{n_k}) = \infty \) is SBC with respect to \( T_E \), then \( f_n \) is SBC with respect to \( T \).

**Proof.** By Birkhoff’s ergodic theorem and Kac’s lemma
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} R_E(T_E^i(x)) = \int E R_E(x) \, d\mu(x) = \frac{1}{\mu(E)} \]
for almost every \( x \). Fix any \( \varepsilon \) with \( 0 < \varepsilon < 1/\mu(E) - 1 \), we can choose \( N_0(x, \varepsilon) \) such that if \( n > N_0 \),
\[ (\frac{1}{\mu(E)} - \varepsilon)n < \sum_{i=0}^{n-1} R_E(T_E^i(x)) < (\frac{1}{\mu(E)} + \varepsilon)n. \quad (2) \]

Let \( g_n = f_{[n(1/\mu(E)+\varepsilon)]} \), where \([t]\) is the smallest integer which is not less than \( t \). Then since \( \{f_n\}_{n} \) is a decreasing sequence, \( \sum_n \mu(g_n) = \infty \). Thus \( \{g_n\}_{n} \) is SBC with respect to \( T_E \) and
\[ \lim_{N \to \infty} \frac{\sum_{1 \leq n < N} g_n \circ T_E^n(x)}{\sum_{1 \leq n < N} \mu(g_n)} = \frac{1}{\mu(E)}, \quad \text{a.e.} \quad (3) \]

Let \( Q(n,x) = \sum_{i=0}^{n-1} R_E(T_E^i(x)) \). Then \( T^{Q(n,x)}(x) = T_E^n(x) \) and by \( (2) \) we have for \( n > N_0 \)
\[ g_n(T_E^n(x)) = f_{[n(1/\mu(E)+\varepsilon)]}(T^{Q(n,x)}(x)) \leq f_{Q(n,x)}(T^{Q(n,x)}(x)). \]
Since $f_k(T^k(x)) = 0$ if $k \neq Q(n, x)$, $n = 1, 2, \ldots$, we have for $Q(N, x) \leq K < Q(N + 1, x)$

$$
\sum_{1 \leq n < N} g_n \circ T^k E^n(x) \leq \sum_{1 \leq k < K} f_k \circ T^k(x) + \sum_{1 \leq n < N} f_1 \circ T^k E^n(x).
$$

Note that $\sup |f_k| \leq M$. By dividing both sides by $\sum_{1 \leq k < K} \mu(f_k)$ we have for $Q(N, x) \leq K < Q(N + 1, x)$

$$
\frac{\sum_{1 \leq k < K} f_k \circ T^k}{\sum_{1 \leq k < K} \mu(f_k)} \geq \frac{\sum_{1 \leq n < N} g_n \circ T^k E^n - \sum_{1 \leq n < N} f_1 \circ T^k E^n}{\sum_{1 \leq n < N} \mu(g_n)} \geq \frac{1}{\sum_{1 \leq n < N} \mu(g_n)} \sum_{1 \leq n < N} \mu(f_1). \quad (4)
$$

By Lemma 5.1 in Appendix, if we put $C = 1/\mu(E) + \varepsilon$ and $a_n = \mu(f_n)$, then for $Q(N, x) \leq K < Q(N + 1, x)$, $N > N_0$ we have

$$
\sum_{1 \leq k < K} \mu(f_k) \leq \sum_{1 \leq k < (1/\mu(E) + \varepsilon)(N + 1)} \mu(f_k) \leq (\frac{1}{\mu(E)} + \varepsilon)(\sum_{1 \leq n < N + 1} \mu(g_n) + \mu(f_1)). \quad (5)
$$

Since $\sum \mu(f_k)$ and $\sum \mu(g_k)$ diverge, by (3) and (5), the inequality (4) implies that

$$
\liminf_{K \to \infty} \frac{\sum_{1 \leq k < K} f_k \circ T^k E^n}{\sum_{1 \leq k < K} \mu(f_k)} \geq \frac{1}{1 + \varepsilon \mu(E)}, \quad \text{a.e.}
$$

For the other direction, let $r_n = f_{\lfloor n(1/\mu(E) - \varepsilon) \rfloor}$, where $\lfloor t \rfloor$ is the largest integer which does not exceed $t$. Then $\sum_n \mu(r_n) = \infty$. Thus $\{r_n\}$ is also SBC with respect to $T_E$ and

$$
\lim_{N \to \infty} \frac{\sum_{1 \leq n < N} r_n \circ T^k E^n(x)}{\sum_{1 \leq n < N} \mu(r_n)} = \frac{1}{\mu(E)}, \quad \text{a.e.} \quad (6)
$$

By Lemma 5.1 in Appendix, if we put $C = 1/\mu(E) - \varepsilon$ and $a_n = \mu(f_n)$, for $Q(N, x) \leq K < Q(N + 1, x)$, $N > N_0$ we have

$$
\sum_{1 \leq k < K} \mu(f_k) \geq \sum_{1 \leq k < (1/\mu(E) - \varepsilon)N} \mu(f_k) \geq (\frac{1}{\mu(E)} - \varepsilon) \sum_{1 \leq n < N} \mu(r_n) - \mu(f_1). \quad (7)
$$

If $n > N_0$, then $r_n(T^n E^n(x)) \geq f_{Q(n, x)}(TQ(n, x)(x))$. Hence for $Q(N, x) \leq K < Q(N + 1, x)$ we have $\sum_{1 \leq n < N} r_n \circ T^k E^n(x) + \sum_{1 \leq n < N_0} f_1 \circ T^k E^n(x) \geq \sum_{1 \leq k < K} f_k \circ T^k(x)$ and

$$
\frac{\sum_{1 \leq k < K} f_k \circ T^k}{\sum_{1 \leq k < K} \mu(f_k)} \leq \frac{\sum_{1 \leq n < N} r_n \circ T^k E^n + \sum_{1 \leq n < N_0} f_1 \circ T^k E^n}{\sum_{1 \leq n < N} \mu(r_n)} \frac{\sum_{1 \leq n < N_0} \mu(r_n)}{\sum_{1 \leq k < K} \mu(f_k)}.
$$

Since $\sum \mu(f_k)$ and $\sum \mu(r_k)$ diverge, by (6) and (7) we have

$$
\limsup_{K \to \infty} \frac{\sum_{1 \leq k < K} f_k \circ T^k(x)}{\sum_{1 \leq k < K} \mu(f_k)} \leq \frac{1}{1 - \varepsilon \mu(E)}, \quad \text{a.e.}
$$

4. A map with an indifferent fixed point. Let $X = [0, 1)$ be the unit interval and $T_α : X \to X$ be the transformation defined by

$$
T_α(x) = \begin{cases} 
  x(1 + 2αxα), & \text{if } 0 \leq x < \frac{1}{2}, \\
  2x - 1, & \text{if } \frac{1}{2} \leq x < 1
\end{cases}
$$
for $0 < \alpha < 1$. Then $T_\alpha$ has an indifferent fixed point at $x = 0$ with $T'_\alpha(0) = 1$. It is well known that $T_\alpha$ has a finite absolutely continuous invariant measure $\mu$ with decreasing density function $h(x) = d\mu/dx$. This map is related with intermittency [11]. Hu [7] showed that
\[
\lim_{x \to 0} x^\alpha h(x) = c
\]
for some constant $c$. Also we have $\mu((0, x)) = x^{1-\alpha}$, i.e., there are positive constants $C_1$ and $C_2$ such that $C_1 x^{1-\alpha} < \mu((0, x)) < C_2 x^{1-\alpha}$ for small $x$ [15]. See also [8] and [16].

**Proposition 4.1.** Let $A_n$ be a decreasing sequence of intervals with $\sum_n \mu(A_n) = \infty$. If $0 \notin \cap_n A_n$, then $A_n$ is SBC with respect to $T_\alpha$ and to the absolute continuous invariant measure $\mu$.

**Proof.** Suppose that $\cap_n A_n$ contains an interval. Let $(a, b)$ be the maximal interval contained in $\cap_n A_n$. Then we can divide $A_n$ into three parts $A_n \cap [0, a]$, $A_n \cap (a, b)$, and $A_n \cap [b, 1]$. The intersections $\cap_n (A_n \cap [0, a])$ and $\cap_n (A_n \cap [b, 1])$ contain no intervals. By the Birkhoff ergodic theorem every sequence of an identical set is SBC, so the sequence of $(a, b)$ is SBC and by Lemma 5.3 (i) we may assume that there is no interval in $\cap_n A_n$.

Let $E = [\frac{1}{2}, 1)$ and $T_E$ be the induced map of $T_\alpha$ on $E$. Then $T_E$ is a mixing piecewise expanding $C^1$ map with a countable partition and $|T'(x)|^{-1}$ is of bounded variation. Thus by Theorem 2.1, every sequence of intervals in $E$ is SBC with respect to $T_E$.

Put $a_0 = 1$, $a_1 = \frac{1}{2}$, and $a_k = (T_\alpha|_{[0, \frac{1}{2})})^{-1}(a_{k-1})$ for $k > 1$ inductively. Then $(T_\alpha)^i(a_k, a_{k-1}) = [a_{k-i}, a_{k-i-1}]$ for each $i \leq k - 1$. Let $E_k = [a_k, 1)$ for $k \geq 1$ and $T_{E_k}$ be the induced map of $T_\alpha$ on $E_k$. Then $T_{E_k}(x) = (T_\alpha)^i(x)$ for $x \in [a_k, a_{k+1}]$, $i > 0$. $T_{E_k}$ is a mixing piecewise expanding $C^1$ map with a countable partition and $|T'(x)|^{-1}$ is of bounded variation. Thus by Theorem 2.1, every sequence of intervals in $E_k$ is SBC with respect to $T_{E_k}$. Since $0 \notin \cap_n A_n$, there is $k$ such that $A_n \subset E_k$ for all large $n$. By Theorem 3.1 $A_n$ is SBC with respect to $T$. □

**Proposition 4.2.** Put $A_n = [0, n^{1/(\alpha - 1)})$ for $n > 1$. Then $A_n$ is not BC with respect to $T_\alpha$ and the absolute continuous invariant measure $\mu$.

**Proof.** Let
\[
J(x) = \{j \geq 0 \mid (T_\alpha)^j(x) \notin A_j \text{ and } (T_\alpha)^{j+1}(x) \in A_{j+1}\}.
\]
Assume that $(T_\alpha)^j(x) \in A_n$ for infinitely many $n$’s for a given point $x$. Then the cardinality of $J(x)$ is infinite or there is an integer $N \geq 0$ such that $(T_\alpha)^j(x) \in A_j$ for all $j \geq N$, which implies that $(T_\alpha)^N(x) = 0$ since $T_\alpha$ is strictly increasing on each $A_n$. Note that there are only countably many $x$’s with $(T_\alpha)^N(x) = 0$ for some $N$.

Let
\[
B_n = \left[ \frac{1}{2}, \frac{1}{2} + \frac{n^{1/(\alpha - 1)}}{2} \right)
\]
for $n > 1$. Then $T_\alpha(B_n) = A_n$, so if $(T_\alpha)^n(x) \in A_n$ for some $n \geq 1$ then $(T_\alpha)^{n-1}(x) \in A_n$ or $(T_\alpha)^{n-1}(x) \in B_n$. Thus, if $(T_\alpha)^n(x) \in A_n$, then either $(T_\alpha)^i(x) \in A_n$ for all $i$, $0 \leq i \leq n$ or there is $\ell$ with $0 \leq \ell < n$ such that $(T_\alpha)^\ell(x) \in B_n$ and $(T_\alpha)^i(x) \in A_n$ for $\ell < i \leq n$. Hence we have
\[
J(x) = \{j \geq 0 \mid (T_\alpha)^j(x) \in B_{j+1}\}.
\]
Since the invariant density function $h(x)$ is decreasing, $\mu(B_n)$ is bounded by

$$\mu(B_n) \leq h \left( \frac{1}{2} \right) \frac{n^{1/(\alpha-1)}}{2}$$

and $\sum_n \mu(B_n) < \infty$. Therefore, by the first Borel-Cantelli lemma, for only finitely many $n$’s $(T_n)^n(x) \in B_{n+1}$ and the cardinality $|J(x)|$ is finite for almost every $x$. Hence for almost every $x$, we have $(T_n)^n(x) \in A_n$ for only finitely many $n$’s. But $\mu(A_n) \approx (n^{1/(\alpha-1)})^{1-\alpha} = n^{-1}$ and $\sum_n \mu(A_n) = \infty$. \hfill \Box

We introduce the SBC property with respect to Lebesgue measure in the following theorem.

**Theorem 4.3.** Let $A_n$ be a decreasing sequence in $X$ with $\sum_n \lambda(A_n) = \infty$. Then $A_n$ is SBC with respect to $T_a$ and Lebesgue measure $\lambda$, in the sense that for almost every $x$

$$\lim_{N \to \infty} \frac{S_N(x)}{\sum_{n=1}^N \lambda(A_n)} = \begin{cases} h(t), & \text{if } \cap_n A_n = \{t\}, \ t \neq 0 \\ \infty, & \text{if } \cap_n A_n = \{0\} \\ \frac{1}{b-a} \int_a^b h(t)dt, & \text{if } \cap_n A_n = [a,b]. \end{cases}$$

**Proof.** It is known [15] that $h(x)$ is continuous on $(0,1]$.

First, assume that $\cap_n A_n = \{t\}, \ t \neq 0$. Then by Proposition 4.1 we have

$$\frac{S_N(x)}{\sum_{n=1}^N \mu(A_n)} \to 1.$$  

Since $\mu(A_n)/\lambda(A_n) \to h(t)$, we have the proof.

Next assume that $\cap_n A_n = \{0\}$. Let $A_n^{(0)} = A_n$, $A_n^{(i)} = (T_a|_{[0,1]}^{-1}A_n, \ i \geq 1$ and $B_n^{(i)} = (T_a|_{[0,1]}^{-1}A_n^{(i)})^{-1}A_n^{(0)}$, $i \geq 0$. Note that for each $i$ the sequence $\{B_n^{(i)}\}$ is SBC with respect to $\lambda$. Then we have

$$T_a^{-n}A_n = B_n^{(0)} \cup B_n^{(n-1)} \cup T_a^{-1}B_n^{(n-2)} \cup \cdots \cup T_a^{-n+1}B_n^{(0)}$$

and the unions are disjoint. Therefore, the number of $n$’s such that $(T_n)^n(x) \in A_n$ for $1 \leq n \leq N$ is

$$S_N(x) = \sum_{n=1}^N 1_{T_a^{-n}A_n}(x) = \sum_{n=1}^N 1_{A_n^{(0)}}(x) + \sum_{k=0}^{N-1} \sum_{n=1}^{N-k} 1_{T_a^{-n+1}B_n^{(k)}}(x).$$

Since $\sum_{n=1}^\infty \lambda(B_n^{(0)}) = \frac{1}{2} \sum_{n=1}^\infty \lambda(A_n) = \infty$ and $(T_a)'(0) = 1$, for any $k \geq 1$ we have

$$\frac{S_N(x)}{\sum_{n=1}^N \lambda(A_n)} = \frac{S_N(x)}{2 \sum_{n=1}^N \lambda(B_n^{(0)})} \geq \frac{\sum_{k=0}^{N-1} \sum_{n=1}^{N-k} 1_{T_a^{-n+1}B_n^{(k)}}(x)}{2 \sum_{n=1}^N \lambda(B_n^{(0)})}$$

and

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N-k} 1_{T_a^{-n+1}B_n^{(k)}}(x)}{\sum_{n=1}^N \lambda(B_n^{(0)})} = \lim_N \frac{\sum_{n=1}^{N-k} 1_{T_a^{-n+1}B_n^{(k)}}(x)}{\sum_{n=1}^{N-k} \lambda(B_n^{(k)})} = \frac{\sum_{n=1}^{N-k} 1_{T_a^{-n+1}B_n^{(k)}}(x)}{\sum_{n=1}^{N-k} \lambda(B_n^{(k)})} = h\left(\frac{1}{2}\right) > 0.$$
Hence we have
\[
\lim_{N \to \infty} \frac{S_N(x)}{\sum_{n=1}^{N} \lambda(A_n)} = \infty.
\]

Now, we assume that \( \cap A_n = [a, b] \). Then for any \( \varepsilon > 0 \) there is \( M > 0 \) such that \((a, b) \subset A_n \subset (a - \varepsilon, b + \varepsilon) \cap X \) for \( n > M \). Hence we have
\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} 1_{(a, b)}(T^n(x))}{(b - a + 2\varepsilon)N} \leq \lim_{N \to \infty} \frac{S_N(x)}{\sum_{n=1}^{N} \lambda(A_n)} \leq \lim_{N \to \infty} \frac{\sum_{n=1}^{N} 1_{(a - \varepsilon, b + \varepsilon) \cap X}(T^n(x))}{(b - a)N}.
\]

By the Birkhoff ergodic theorem for almost every \( x \)
\[
\frac{\mu((a, b))}{b - a + 2\varepsilon} \leq \lim_{N \to \infty} \frac{S_N(x)}{\sum_{n=1}^{N} \lambda(A_n)} \leq \frac{\mu((a - \varepsilon, b + \varepsilon) \cap X)}{b - a}.
\]

Since \( \mu \) is an absolutely continuous measure, we have
\[
\lim_{N \to \infty} \frac{S_N(x)}{\sum_{n=1}^{N} \lambda(A_n)} = \frac{\mu((a, b))}{b - a}.
\]

\[\square\]

Let \( \xi = \{[0, \frac{1}{\ell}], (\frac{1}{\ell}, 1)\} \) be a partition of \( X \) and \( \xi_n = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi \), where \( \xi \vee \eta = \{A \cap B : A \in \xi, B \in \eta\} \). In [7], Hu showed that there exist constants \( C_1 > 0 \) and \( \ell \) such that for any \( m \geq 0 \) and \( E \in \xi_m \) and for any measurable set \( F \subset [\frac{1}{\ell}, 1) \),
\[
|\mu(T^{-n-m}F \cap E) - \mu(F)\mu(E)| \leq \frac{C_1m^{\beta-1}}{(n - \ell)^{\beta-1}}\mu(F)\mu(E), \quad \text{for any } n \geq \ell,
\]
where \( \beta = 1/\alpha \). See also [10] and [16].

**Theorem 4.4.** Suppose that \( \alpha < \frac{3 - \sqrt{5}}{2} \). Let \( A_n \) be a sequence of intervals in \([d, 1) \), \( d > 0 \) with \( \sum \mu(A_n) = \infty \). Then \( A_n \) is SBC with respect to \( T_n \) and to the absolutely continuous invariant measure \( \mu \).

**Proof.** Let \( E \) be an interval in \( X \) and \( F \) be a measurable set in \([\frac{1}{\ell}, 1) \). Then there are \( E_1 \) and \( E_2 \) in \( \xi_m \) such that \( E_1 \subset E \subset E_2 \) and \( \mu(E_2 \setminus E_1) \leq C_0m^{-\beta} \) (see [16]). By (8) we have
\[
\mu(E_1) + \frac{C_1m^{\beta-1}}{(n - \ell)^{\beta-1}}\mu(E_1) \leq \frac{\mu(T^{-n-m}F \cap E)}{\mu(F)} \leq \mu(E_2) + \frac{C_1m^{\beta-1}}{(n - \ell)^{\beta-1}}\mu(E_2)
\]
so we have
\[
\left| \frac{\mu(T^{-n-m}F \cap E)}{\mu(F)} - \mu(E) \right| \leq C_0m^{-\beta} + \frac{C_1m^{\beta-1}}{(n - \ell)^{\beta-1}}.
\]

Put \( p = \frac{2\beta-1}{\beta-1} \). If \( m^p \leq n \leq (m + 1)^p \), then for \( m \geq \ell \)
\[
\left| \frac{\mu(T^{-n-m}F \cap E)}{\mu(F)} - \mu(E) \right| \leq C(m - 1)^{-\beta},
\]
where \( C = C_0 + C_1 \). Thus, we have
\[
\sum_{k=0}^{\infty} \left| \frac{\mu(T^{-k}F \cap E)}{\mu(F)} - \mu(E) \right| \leq \ell^p + \ell + \sum_{m=\ell}^{\infty} \sum_{n=[m^p]}^{(m+1)^p} \left| \frac{\mu(T^{-n-m}F \cap E)}{\mu(F)} - \mu(E) \right|
\]
\[
\leq \ell^p + \ell + C \sum_{m=\ell}^{\infty} ((m+1)^p - m^p + 2)(m - 1)^{-\beta}.
\]
Let
\[ D = \ell^p + \ell + C \sum_{m=\ell}^{\infty} (p(m + 1)^{p-1} + 2)(m - 1)^{-\beta}. \]

If \( \beta > \frac{3+p}{2} \), then \( D < \infty \) and
\[ \sum_{k=0}^{\infty} \left| \mu(T^{-k} F \cap E) - \mu(F)\mu(E) \right| \leq D\mu(F). \]

Hence, by Lemma 1.1 we have the proof for a sequence of intervals \( A_n \) in \([1/2, 1)\).

Put \( a_0 = \frac{1}{2} \) and \( a_k = (T_{\alpha} |_{[0, \frac{1}{2}]})^{-1}(a_{k-1}) \) for \( k \geq 1 \) inductively. Let \( B = \left[ \frac{1}{2}, 1 \right) \).

If \( A_n \subset [a_1, a_0) \), then \( T(A_n) \subset B \) so \( \{T(A_n)\}_n \) and \( \{T^{-1} \circ T(A_n) \cap B\}_n \) are SBC. Also \( \{T^{-1} \circ T(A_n)\}_n \) is SBC by Lemma 5.2. By Lemma 5.3 (ii) \( A_n \) is SBC because \( A_n = T^{-1} \circ T(A_n) \setminus (T^{-1} \circ T(A_n) \cap B) \) and \( \mu(T^{-1} \circ T(A_n) \cap B) < c\mu(T^{-1} \circ T(A_n)) \) for some constant \( c \). Now if \( A_n \subset [a_1, 1) \), then \( A_n \) is SBC by Lemma 5.3 (i). Inductively, if \( A_n \subset [a_k, 1) \) for some \( k \), then \( A_n \) is SBC.

5. Appendix.

**Lemma 5.1.** Let \( \{a_n\}_{n=1}^{\infty} \) be a nonnegative decreasing sequence and \( C > 1 \). Then for any integer \( N > 0 \) we have

\[ C \sum_{1 \leq n < N} a_{\lfloor Cn \rfloor} - a_1 \leq \sum_{1 \leq k < CN} a_k \leq C \left( \sum_{1 \leq n < N} a_{\lfloor Cn \rfloor} + a_1 \right). \]

**Proof.** Put \( a_0 = a_1 \) for convenience. (i) Let \( f(n + t) = a_{n+1} \) for an integer \( n \geq 0 \), \( 0 \leq t < 1 \). Then we have
\[ f_0^{[CN]} f(x) dx = \sum_{1 \leq n < CN} a_n. \]
Let \( g(C(n+t)) = f(Cn) \) for an integer \( n \geq 0 \), \( 0 \leq t < 1 \). Then we have
\[ g(x) \geq f(x) \quad \text{and} \quad f_0^{CN} g(x) dx = C \sum_{0 \leq n < N} a_{\lfloor Cn \rfloor}. \]

(ii) Let \( f(n + t) = a_n \) for an integer \( n \geq 0 \), \( 0 \leq t < 1 \). Then
\[ f_0^{[CN]} f(x) dx = \sum_{0 \leq n < CN} a_n. \]
Let \( g(C(n+t)) = f(C(n+1)) \) for an integer \( n \geq 0 \), \( 0 \leq t < 1 \). Then
\[ g(x) \leq f(x) \quad \text{and} \quad f_0^{CN} g(x) dx = C \sum_{1 \leq n < N} a_{\lfloor Cn \rfloor}. \]

**Lemma 5.2.** \( A_n \) is SBC if and only if \( T^{-1} A_n \) is SBC.

**Proof.** Let \( E_N \) be the number of \( n \)'s such that \( T^n(x) \in A_n, 1 \leq n \leq N \) and \( S_N = \sum_{n=1}^{N} \mu(A_n) \). Let \( E \) be the set of point such that \( E_N/S_N \to 1 \). Then \( T^{-1} E \) is the set of point such that \( E_N/S_N \to 1 \) for the sequence \( T^{-1} A_n \).

**Lemma 5.3.** (i) Let \( A_n \) and \( B_n \) be SBC with \( A_n \cap B_n = \emptyset \). Then \( B_n \cup A_n \) is SBC.

(ii) Let \( A_n \) and \( B_n \) be SBC with \( A_n \subset B_n \). If \( \sum_{n=1}^{N} \mu(A_n) < c \sum_{n=1}^{N} \mu(B_n) \) for some constant \( 0 < c < 1 \), then \( B_n \setminus A_n \) is SBC.

**Proof.** Let \( E_N \) and \( E'_N \) be the number of \( n \)'s such that \( T^n(x) \in A_n, 1 \leq n \leq N \) and \( T^n(x) \in B_n, 1 \leq n \leq N \) respectively. Let \( S_N = \sum_{n=1}^{N} \mu(A_n) \) and \( S'_N = \sum_{n=1}^{N} \mu(B_n) \). Then for every \( \varepsilon \) there is \( M \) such that if \( N > M \)
\[ \left| \frac{E_N}{S_N} - 1 \right| < \varepsilon \quad \text{and} \quad \left| \frac{E'_N}{S'_N} - 1 \right| < \varepsilon. \]

Hence for \( N > M \)
\[ \left| \frac{E_N + E'_N}{S_N + S'_N} - 1 \right| \leq \left| \frac{E_N}{S_N} - 1 \right| \left| \frac{S_N}{S_N + S'_N} \right| + \left| \frac{E'_N}{S'_N} - 1 \right| \left| \frac{S'_N}{S_N + S'_N} \right| < \varepsilon \left| \frac{S_N + S'_N}{S_N + S'_N} \right| = \varepsilon. \]
and

\[
\frac{E_N - E_N}{S_N - S_N} - 1 \leq \frac{E_N - 1}{S_N - S_N} \frac{S_N}{S'_N - S_N} + \frac{E'_N - 1}{S'_N - S_N} < \epsilon \frac{S_N + S_N}{S_N - S_N} < \frac{1 + c}{1 - c}.
\]

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