Asymptotic Cycles
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ASYMPTOTIC CYCLES*

By Sol Schwartzman

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1. Introduction [4]

Let $X$ be a topological space, $G$ a topological group with identity element $e$. Let $f$ be a continuous mapping from $X \times G$ into $X$. For simplicity we will write $f(x, g)$ as $xg$. If $(xg_1)g_2 = x(g_1g_2)$, and if $xe = x$, $G$ is said to act as a topological transformation group on $X$. Such transformation groups have been studied from many points of view. In this paper we will be concerned with the viewpoint first employed systematically by George David Birkhoff. Here the concrete model to be borne in mind is the flow arising from a system of ordinary differential equations on a manifold, and in fact we will confine ourselves to the case where $G$ is the additive group $R$ of the real line with the usual topology.

There is a strong connection between topological dynamics, which is the field we are concerned with here, and ergodic theory, another field in which Birkhoff pioneered. Thus, in analogy with the Poincaré recurrence theorem there are theorems concerning various kinds of topological recurrence. In addition, various incompressibility properties have been studied. These of course are a kind of substitute for the measure-preserving property usually required for transformations in ergodic theory. Other parallels exist—for example, sets of the first category play much the same role in some theorems that one might expect from sets of measure zero. Results of all these types can be found together with many other theorems in the book by Gottschalk and Hedlund cited above.

The close parallelism between the two fields has tended to minimize the attention given to questions which strongly involve the topology on $X$. In particular algebraic topology has played no part in the development of the general theorems in topological dynamics. In this paper a first attempt is made to overcome this restriction, at least to the limited extent of trying to show the role played by the first Betti group. Since the orbits are one-dimensional, it is not surprising that the one-dimensional group plays a leading role. It turns out that, except for a set of orbits having measure zero with respect to every invariant measure, we can associate with each orbit an element of the first Betti group of $X$. We call this the asymptotic cycle of the orbit. In a sense it tells

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how the orbit winds around the space $X$. In the case of flows on the torus, the coefficients (with respect to the natural basis) of the asymptotic cycle associated with an orbit are precisely the winding numbers of Poincaré. We shall see that the study of the asymptotic cycles of orbits is intimately connected with the question of the existence of surfaces of section, and the equally standard problem of finding eigenfunctions of the flow.

Theorems in topological dynamics yield as special cases theorems about differential equations on manifolds. Thus far, apart from consequences of the fact that a Hamiltonian system has an invariant measure which is positive for open sets, there is comparatively little in the general theory that would enable one to use the form of the differential equations to deduce additional information about the asymptotic behavior of the solutions. We will see below however that once the concept of the asymptotic cycle has been introduced we will be able to perform explicit computations, particularly for Hamiltonian systems, which will tell us how an “average” orbit winds around the space.

\section{Kryloff-Bogoliouboff theory \cite{5}}

In addition to the parallel development that has taken place between ergodic theory and topological dynamics, methods have been developed which use results in one field to get theorems in the other. For purposes of future reference we collect here a number of well-known definitions and theorems of this type. A complete exposition which contains proofs of these results and many others will be found in the article by Oxtoby cited above. Here and in all later paragraphs $X$ will denote a compact metric space on which the real line acts as a topological transformation group.

\textbf{Definition.} A positive measure is a measure $\mu$ defined for all Borel sets $S \subseteq X$ such that $\mu(S) \geq 0$ for all $S$. A normalized measure is a positive measure for which $\mu(X) = 1$.

\textbf{Definition.} A measure $\mu$ defined on the Borel sets of $X$ is called invariant provided that for every Borel set $S$ and every real number $t$, $\mu(St) = \mu(S)$.

\textbf{Definition.} A point $p$ in $X$ is called quasi-regular provided that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(pt) \, dt$$

exists for every real-valued continuous function $f(x)$ defined on $X$.

\textbf{Theorem.} For every quasi-regular point $p$ there exists a unique finite measure $\mu_p$ such that for every continuous $f(x)$, $\lim_{T \to \infty} \frac{1}{T} \int_0^T f(pt) \, dt = \int_X f(x) \, d\mu_p(x)$. Moreover, $\mu_p$ is a positive invariant measure such that $\mu_p(X) = 1$.

\textbf{Theorem.} The set of points which are not quasi-regular has measure zero with respect to every finite invariant measure.

\textbf{Definition.} A function $f(x)$ is said to be differentiable with respect to the flow provided that $\lim_{\Delta t \to 0} (f(x \Delta t) - f(x))/\Delta t$ exists uniformly over $X$. This limit will be denoted by $f'(x)$ or $(d/(dt))f(x)$.
The following theorem was communicated orally to the author by Professor Kakutani. It is this theorem that will enable us to state our results for arbitrary compact metric spaces, without restricting ourselves to differentiable manifolds.

**Theorem (Kakutani).** Any continuous function \( f(x) \) on \( X \) can be approximated uniformly by functions which are differentiable with respect to the flow.

**Proof.** Let \( g_\varepsilon(x) = 1/\varepsilon \int_0^\varepsilon f(xt) \, dt \). Then

\[
\frac{g_\varepsilon(x\Delta t) - g_\varepsilon(x)}{\Delta t} = \frac{1}{\varepsilon\Delta t} \left( \int_{\varepsilon}^{\varepsilon + \Delta t} f(xt) \, dt - \int_{0}^{\varepsilon} f(xt) \, dt \right)
\]

\[
= \frac{1}{\varepsilon} \left( \frac{1}{\Delta t} \int_{\varepsilon}^{\varepsilon + \Delta t} f(xt) \, dt - \frac{1}{\Delta t} \int_{0}^{\Delta t} f(xt) \, dt \right).
\]

Since, as \( t \) goes to \( t_0 \), \( f(xt) \) approaches \( f(xt_0) \) uniformly, it follows that \( g_\varepsilon(x) \) is differentiable with respect to the flow, and in fact \( g_\varepsilon'(x) = (1/\varepsilon)(f(x\varepsilon) - f(x)) \). It is evident that as \( \varepsilon \) goes to zero, \( g_\varepsilon(x) \) approaches \( f(x) \) uniformly, so the theorem is proved.

3. The theory of Eilenberg and Bruschiinsky [1], [3]

Because of the relation between quasi-regular points and continuous functions, the theory referred to in the title of this section enables us to connect properties of the flow with the first Betti group of the space. We will summarize those results which will be needed later on, referring the reader to the articles cited above for proofs. As in the rest of this paper, the space \( X \) referred to below is assumed to be compact metric.

Throughout the rest of this paper the homology and cohomology groups we refer to are those obtained from the Čech theory using real coefficients. When we speak of an integral cycle we will mean an element of this homology group which arises from integral cycles in the ordinary sense when we represent our space as an inverse projective limit of polyhedra. By a cocycle with integral periods we will mean an element of the cohomology group which, when considered as a functional on the homology group, assigns integer values to each integral cycle. Actually we will use only the one-dimensional groups. It should be borne in mind that any closed parametrized curve \( K \) in \( X \) determines an element of the first homology group in the Čech theory, since a map of \( X \) into a polyhedron sends \( K \) into a curve of the polyhedron.

**Definition.** The set of all continuous complex-valued functions of absolute value one defined on \( X \) will be denoted by \( C(X) \). The subset of \( C(X) \) consisting of those functions which can be written in the form \( \exp(2\pi iH(x)) \), where \( H(x) \) is continuous and real valued, will be denoted by \( R(X) \). If \( K \) is any parametrized curve in \( X \) and \( f(x) \in C(X) \), \( H_\kappa \arg f(x) \) will mean the change in the angular variable for \( f(x) \) along \( K \).

Notice that when \( f(x) = \exp(2\pi iH(x)), (1/2\pi)H_\kappa \arg f(x) = H(p_2) - H(p_1) \), where \( p_1 \) and \( p_2 \) are the initial and terminal points of \( K \).
Theorem. If \( f(x) \in C(X) \) and \( K \) is a closed parametrized curve, \( (1/2\pi) \Delta \arg f \) is an integer depending only on the element of the first homology group corresponding to \( K \) and the coset determined by \( f(x) \) in \( C(X)/R(X) \).

It is obvious that if \( X \) were a polyhedron the above theorem would immediately enable us to associate with each element of \( C(X)/R(X) \) an element of the first cohomology group which, as a functional on the homology group, would assign to each cycle \( K \) the integer mentioned in the above theorem. It turns out to be possible to generalize this procedure in a natural way so that it applies to any compact metric space \( X \).

Theorem. The mapping indicated above, sending \( C(X)/R(X) \) into the first cohomology group, is an algebraic isomorphism onto all \( \check{C} \)ech cocycles with integral periods.

Theorem. If \( f_1(x) \) and \( f_2(x) \) belong to \( C(X) \) and \( |f_1(x) - f_2(x)| < 1/2 \) then these two functions determine the same element mod \( R(X) \).

Finally, suppose \( X \) is a differentiable manifold and \( f(x) \in C(X) \) is a differentiable function. In a suitable local coordinate system \( f(x) = \exp(2\pi i H(x)) \) where \( H(x) \) is differentiable and uniquely determined up to an additive constant. If \( H_1 \) and \( H_2 \) are determined as above in two different coordinate systems, \( dH_1 \) and \( dH_2 \) agree in the region of overlapping. Thus \( f(x) \) determines a closed one-form over \( X \). This mapping, sending the differentiable elements of \( C(X) \) into closed one-forms, induces the natural mapping of \( C(X)/R(X) \) onto the elements with integral periods in the quotient group of closed one-forms modulo bounding one-forms.

4. Asymptotic cycles

Definition. Let \( f(x) \) be an element of \( C(X) \) and let \( p \) be any point in \( X \). By \( \Delta_{(p, pt)} \arg f \) we mean the change in the angular variable of \( f(x) \) along the orbit going from \( p \) to \( pt \).

Lemma. Let \( p \) be any point of \( X \) and suppose \( f_1(x) \) and \( f_2(x) \) are two elements of \( C(X) \) which determine the same element of \( C(X)/R(X) \). Then

\[
(1/2\pi) \Delta_{(p, pt)} \arg f_1 - (1/2\pi) \Delta_{(p, pt)} \arg f_2 = 0(1).
\]

Proof. We know that for some continuous \( H(x) \), \( f_1(x) = f_2(x) \exp(2\pi i H(x)) \). Hence \( (1/2\pi) \Delta_{(p, pt)} \arg f_1 - (1/2\pi) \Delta_{(p, pt)} \arg f_2 = H(pt) - H(p) \). Since \( H(x) \) must be bounded, the lemma is proved.

Theorem. Let \( p \) be any quasi-regular point. Then for any \( f(x) \in C(X) \)

\[
\lim_{t \to \infty} (1/(2\pi t)) \Delta_{(p, pt)} \arg f \text{ exists. Moreover, this limit depends only on the element of } C(X)/R(X) \text{ determined by } f(x). \text{ The induced mapping of } C(X)/R(X) \text{ into the real line is a group homomorphism.}
\]

Proof. By the above lemma, if this limit exists for any \( f_1(x) \) in \( C(X) \) it exists and has the same value for any \( f_2(x) \) equivalent to \( f_1(x) \mod R(X) \). Given any \( f_2(x) \) in \( C(X) \) we know we can approximate \( f_2(x) \) as closely as we wish by a function \( f_1(x) \) in \( C(X) \) which is differentiable with respect to the flow. (A trivial supplement to Kakutani’s theorem in Section two shows this.) If we choose
$f_1(x)$ so that $|f_1(x) - f_2(x)| < \frac{1}{2}$, then $f_1(x)$ and $f_2(x)$ are equivalent mod $R(X)$. Since $f_1(x)$ is differentiable it is easy to see that

$$(1/(2\pi T))\Delta_{(p, pt)} \arg f_1 = 1/(2\pi i T) \int_0^T (f'_1(pt))/(f_1(pt)) \, dt.$$ 

Since $p$ is assumed to be quasi-regular, this approaches a limit as $T$ goes to infinity. All we need to complete the proof of the theorem is the fact that if $f_1(x)$ and $f_2(x)$ belong to $C(X)$

$$\lim_{T \to \infty} (1/(2\pi T))\Delta_{(p, pt)} \arg f_1 = \lim_{T \to \infty} (1/(2\pi T))\Delta_{(p, pt)} \arg f_2.$$ 

This, however, is obvious.

**Corollary.** If $p$ is quasi-regular and $f(x) \in C(X)$, there is a constant $\lambda$ such that

$$(1/2\pi)\Delta_{(p, pt)} \arg f = \lambda t + o(t).$$

Thus we have associated with each quasi-regular point $p$ a homomorphism of $C(X)/R(X)$ into the real numbers. Since $C(X)/R(X)$ may be identified with the elements of the first cohomology group with integral periods, and since every element of the first cohomology group with real coefficients is always expressible as a finite linear combination of these, we can uniquely extend this homomorphism to a linear functional $A_p$ on the first cohomology group with real coefficients.

For convenience all measures $\mu$ will henceforth be assumed to be normalized.

We do this because we will want to think of $\int_X f(x) \, d\mu(x)$ as an average value, without introducing the coefficient $1/(\mu(X))$. Notice that for any quasi-regular point $p$, $\mu_p$ is a normalized measure.

**Theorem.** If $\mu$ is a normalized invariant measure and $f(x) \in C(X)$ then

$$\int_X A_p[f] \, d\mu(p)$$

exists and depends only on the equivalence class mod $R(X)$ to which $f(x)$ belongs. Moreover if $f(x)$ is differentiable with respect to the flow,

$$\int_X A_p[f] \, d\mu(p) = 1/(2\pi i) \int_X (f'(p))/(f(p)) \, d\mu(p).$$

**Proof.** It is only necessary to prove the second part of the theorem.

$$\int_X A_p[f] \, d\mu(p) = \int_X \left( \lim_{T \to \infty} \frac{1}{2\pi i T} \int_0^T f'(pt)/(f(pt)) \, dt \right) d\mu(p).$$

Since the quantity inside the parentheses is uniformly bounded and $\mu$ is a finite measure, this equals

$$\lim_{T \to \infty} \frac{1}{2\pi i T} \int_0^T f'(pt)/(f(pt)) \, dt \, d\mu(p).$$
which by Fubini’s theorem equals
\[ \lim_{T \to \infty} \frac{1}{(2\pi i T)} \int_0^T \int_X (f'(pt))/(f(pt)) \, d\mu(p) \, dt. \]
Since we are assuming that \( \mu \) is an invariant measure,
\[ \int_X (f'(pt))/(f(pt)) \, d\mu(p) = \int_X (f'(p))/(f(p)) \, d\mu(p). \]
Using this to simplify the above expression we get the desired result.

Thus the expression \( 1/(2\pi i) \int_X (f'(p))/(f(p)) \, d\mu(p) \) determines a linear functional \( A_\mu \) on the first co-Betti group which is the average with respect to the measure \( \mu \) of the functionals \( A_p \). Notice that if \( p \) is any quasi-regular point, then \( A_p = A_{\mu_p} \). Thus, if \( \mu_{p_1} = \mu_{p_2} \) then \( A_{p_1} = A_{p_2} \). In addition we note that if \( A_p \) is the same for all quasi-regular points \( p \) then \( A_p \) can be defined for every point of \( X \), even those which are not quasi-regular, and will be the same as for the quasi-regular points. This follows from the corresponding theorem on time averages of continuous functions. We omit the details. If we assume that the first Betti number of \( X \) is finite, a linear functional on the first cohomology group corresponds to an element of the first homology group. We will continue to use \( A_p \) and \( A_\mu \) to denote the elements in the first homology group corresponding to the functionals \( A_p \) and \( A_\mu \), and we will refer to these elements as the \( (p) \)-asymptotic cycle and the \( (\mu) \)-asymptotic cycle. The following theorem gives the geometric interpretation.

**Theorem.** Let \( p \) be any quasi-regular point. For each \( t \) let \( K_t \) be some parametrized curve going from \( pt \) to \( p \). Suppose that all the curves \( K_t \) are parametrized equi-uniformly continuously from the interval \([0, 1]\). Let \( C_t \) be the curve determined by the orbit from \( p \) to \( pt \) followed by the curve \( K_t \), and let \( \tilde{C}_t \) be the corresponding element of the first Betti group. Then \( \lim_{t \to \infty} \tilde{C}_t/t = A_p \).

If \( X \) is a Riemannian manifold the restriction of equiuniform continuity can be met by letting the curves \( K_t \) be uniformly bounded in length. The purpose of the restriction is to limit, to an ultimately negligible amount, the extent to which the curves \( K_t \) can wander about the space.

**Proof.** Since we are assuming that the first Betti number of \( X \) is finite it is enough to prove convergence for every linear functional on the first Betti group, and in order to do this it is sufficient to show that for every \( f(x) \) in \( C(X) \),
\[ A_p(f) = \lim_{t \to \infty} (1/(2\pi t)) \Delta_{C_t} \arg f. \]
Since by definition \( A_p(f) = \lim_{t \to \infty} (1/(2\pi t)) \Delta_{(p, pt)} \arg f \) it is sufficient to show that \( \lim_{t \to \infty} (1/(2\pi t)) \Delta_{K_t} \arg f = 0 \). This, however, is an immediate consequence of the fact that the arcs \( K_t \) can be parametrized equi-uniformly continuously.

Thus we have shown that if \( C_t \) is obtained by closing up the orbit from \( p \) to \( pt \) by means of a “short” arc, then \( \tilde{C}_t \) is in a sense asymptotic to \( tA_p \). The geometric interpretation for \( A_\mu \) is evident; if we regard \( A_p \) as a function from
X to the first homology group of X, $A_\mu$ is the average of this vector valued function and therefore describes how an “average” orbit wanders around X. To determine $A_\mu$ for a given flow, it is necessary to choose any convenient set of functions $f_1(x), \ldots, f_k(x)$ corresponding to a basis for the first cohomology group and compute $1/(2\pi i) \int_X (f'_i(p))/(f_i(p)) \, d\mu(p)$ for $1 \leq i \leq k$. Note that in the case of a flow defined by a vector field on a manifold, this integrand can be computed explicitly whether or not the differential equations can be solved explicitly. These constants determine $A_\mu$ as a functional on the first cohomology group and therefore determine $A_\mu$ completely. We will carry this idea further in the section below devoted to Hamiltonian systems.

5. Eigenfunctions

Definition. A function $f(x)$ in $C(X)$ is said to be an eigenfunction with respect to the flow $f(pt) = \exp (2\pi i \lambda t)f(p)$ for some real number $\lambda$. We call $\lambda$ the eigenvalue associated with $f(x)$.

Theorem. If $f(x)$ is any eigenfunction and $\mu$ is any invariant measure then $A_\mu(f)$ equals the eigenvalue associated with $f(x)$.

Proof. This follows immediately from the formula

$$A_\mu(f) = 1/(2\pi i) \int (f'(x))/(f(x)) \, d\mu(x).$$

Corollary. Two eigenfunctions which belong to the same equivalence class mod $R(X)$ have the same eigenvalue.

This follows from the well-known fact that there always exists at least one invariant measure $\mu$.

It follows from the above theorem that if $f_1, \ldots, f_k$ correspond to a set of generators for the first cohomology group, the subgroup of the real line generated by $A_\mu(f_1), \ldots, A_\mu(f_k)$ contains all eigenvalues arising from continuous eigenfunctions.

Thus for any $f(x)$ in $C(X)$, $A_\mu(f)$ may be thought of as a generalized eigenvalue associated with the element of $C(X)/R(X)$ determined by $f(x)$. This is particularly tempting since, for any quasi-regular point $p$, if we let $\lambda = A_\mu(f)$; $(1/2\pi)\Delta_{(p, pt)} \arg f = \lambda t + o(t)$. Thus $A_\mu(f)$ represents the number of times per unit of time that an “average” orbit gets wrapped around the unit circle by the function $f(x)$. However this viewpoint cannot be pushed too far. It is possible to show by examples that a generalized eigenvalue may exist without there being even a measurable eigenfunction corresponding to it. Conversely, examples can be given of measurable eigenfunctions which do not correspond to any generalized eigenvalues.

6. Spectrally determinate flows

Definition. The flow in $X$ is said to be spectrally determinate provided $A_\mu$ is independent of $\mu$. 

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Obviously, in order for a flow to be spectrally determinate it is necessary and sufficient that the generalized eigenvalue associated with each element of $C(X)/R(X)$ be independent of $\mu$. Another equivalent condition would be to require that $A_\mu$ be the same for every $p$.

It is well known that a measure $\mu$ is invariant with respect to a flow if and only if $\int_X (f(xt) - f(x)) \, d\mu(x) = 0$ for every real-valued continuous function $f(x)$ and every $t$. Since every continuous function can be uniformly approximated by functions which are differentiable with respect to the flow it is sufficient to require that the above equality hold for such functions.

**Definition.** Let $D$ be the set of functions $g(x)$ such that $g(x) = f'(x)$ for some real-valued function $f(x)$ which is differentiable with respect to the flow. Let $\bar{D}$ be the closure of $D$ when we put the topology of uniform convergence on the space of real-valued continuous functions on $X$.

We recall some results of a standard nature concerning invariant measures. Anyone familiar, for example, with the paper by Oxtoby cited in the bibliography should have no difficulty supplying those details of proof which we omit.

**Theorem.** The closure of the set of functions of the form $f(xt) - f(x)$ in the topology of uniform convergence is identical with $\bar{D}$. (Here $f(x)$ ranges over the space of all continuous functions on $X$ and $t$ ranges over all real values.)

**Proof.** If $g \in D$, then for some continuous function $f(x)$,

$$g(x) = \lim_{\Delta t \to 0} \left( (1/\Delta t)f(x\Delta t) - (1/\Delta t)f(x) \right).$$

This proves inclusion in one direction. On the other hand, if $f(x)$ is differentiable with respect to the flow, $f'(x) = g(x)$ belongs to $D$. For any real number $T$,

$$f(xT) - f(x) = \int_0^T g(xt) \, dt.$$

If we make use of Riemann sums to approximate this integral it is easy to show that the approximation is uniform over $X$. The fact that each of the approximating sums belongs to $D$ follows from the observation that $D$ is invariant under translation, i.e., $g(xt_0) = (d/(dt))f(xt_0)$. Thus if $f(x)$ is differentiable with respect to the flow and $T$ is any real number, $f(xT) - f(x)$ belongs to $\bar{D}$. Since any continuous function can be uniformly approximated by functions differentiable with respect to the flow, this suffices to prove the theorem.

**Theorem.** A measure $\mu$ is invariant if and only if $\int_X g(x) \, d\mu(x) = 0$ for every $g(x)$ in $\bar{D}$.

**Theorem.** A function $g(x)$ belongs to $\bar{D}$ if and only if $\int_X g(x) \, d\mu(x) = 0$ for every invariant measure $\mu$.

This theorem is still true if we only require that the above equality hold for every normalized positive invariant measure $\mu$.

**Theorem.** A necessary and sufficient condition that a flow be spectrally de-
terminate is that for every \( f(x) \) in \( C(X) \) which is differentiable with respect to the flow, \( (f'(x)/(2\pi i f(x))) \) must equal a constant plus an element of \( \mathcal{D} \).

**Proof.** We know that the flow is spectrally determinate if and only if for every \( f(x) \) in \( C(X) \) which is differentiable with respect to the flow, there is a constant \( C_f \) depending only on \( f(x) \) such that

\[
\frac{1}{2\pi i} \int_x (f'(x))/(f(x)) \, d\mu(x) = C_f
\]

for every positive normalized invariant measure \( \mu \). This is the same as saying that the integral with respect to every such measure of \( (f'(x))/(2\pi i f(x)) - C_f \) equals zero. Thus our present theorem is a corollary of the preceding one.

Obviously, in order for the conditions of the theorem to be met it is only necessary to show that they are satisfied for any set of functions \( f_1(x) \cdots f_k(x) \) corresponding to a set of generators of \( C(X)/R(X) \). This condition would probably be extremely difficult to apply in many special cases, and it would be very interesting if it were possible to derive a more workable criterion.

If \( x \) is an invariant point it is certainly quasi-regular and \( A_{\mu_x} = A_x = 0 \). If \( x \) is periodic with period \( \tau \) and we let \( C_x \) denote the element of the first Betti group determined by the orbit of \( x \), then \( x \) is quasi-regular and \( A_x = A_{\mu_x} = C_x/\tau \). The next theorem follows immediately from these observations.

**Theorem.** Suppose the flow in \( X \) is spectrally determinate. If there exists either a fixed point or a closed orbit homologous to zero then all closed orbits are homologous to zero. In the remaining case, if \( C_1 \) and \( C_2 \) are closed orbits with periods \( \tau_1 \) and \( \tau_2 \), then \( C_1/\tau_1 \) and \( C_2/\tau_2 \) are homologous. Since \( C_1 \) and \( C_2 \) are integral elements of the first Betti group, it follows in this case that the ratio of the periods of any two closed orbits must be rational. Consequently for any continuous family of periodic orbits, all orbits have the same period. By a continuous family of periodic orbits we mean a one-parameter family that can be parametrized in the natural way by a cylinder.

An interesting example of spectral determinacy occurs on the two dimensional torus. In fact, let \( (dx)/(dt) = a(x, y), \ (dy)/(dt) = b(x, y) \) be a system of differential equations in the plane with \( a(x, y) \) and \( b(x, y) \) assumed to be of class \( C^2 \) and \( a(x, y) \neq 0 \) for all \( x \) and \( y \). If we further assume that \( a(x, y) \) and \( b(x, y) \) are doubly periodic with period one for both \( x \) and \( y \) we may regard these as differential equations on the torus. It is a well known result of Poincaré's that in this case, provided there is no closed orbit, there exist constants \( \lambda \) and \( \mu \) (called the winding numbers) such that if \( x = x(t); y = y(t) \) is any solution curve, \( \lim_{t \to \infty} (x(T) - x(0))/T = \lambda \) and \( \lim_{t \to \infty} (y(T) - y(0))/T = \mu \). If we let \( g_x \) and \( g_y \) be the fundamental cycles associated with \( x \) and \( y \) in the obvious way, this simply says that the asymptotic cycle is defined for every point and equals \( \lambda g_x + \mu g_y \). One interpretation for the asymptotic cycle \( A_F \) is that it (or at any rate its coefficients with respect to a basis for the first Betti group) constitutes the generalization of the winding numbers to arbitrary flows.

There is one case in which we can be certain that the flow is spectrally determinate. Recall that a flow is said to be recurrent provided that there exists a
sequence of real numbers \( \{t_i\} \) such that \( \lim_{i \to \infty} t_i = +\infty \) and
\[
\lim_{i \to \infty} \sup_{x \in X} \rho(x, x t_i) = 0.
\]

**Theorem.** If a recurrent flow is given in an arcwise connected continuum \( X \), the flow is spectrally determinate.

**Proof.** Let \( x_1 \) and \( x_2 \) be any two quasi-regular points in \( X \) and let \( f(x) \) be any function in \( C(X) \). Choose a parametrized arc \( K \) starting at \( x_1 \) and ending at \( x_2 \). Since \( f(x) \) is uniformly continuous there exists an \( \varepsilon > 0 \) such that any arc in \( X \) of diameter less than \( \varepsilon \) has the property that the total change in the angular variable for \( f(x) \) over that arc is less than one radian. Divide \( K \) into subarcs \( K_1, \ldots, K_n \) such that each subarc has diameter less than \( \varepsilon/2 \). For any real number \( t \), \( K, t \) will mean the arc that \( K, t \) goes into after time \( t \) has elapsed.

Since we are assuming the flow is recurrent, there exists a sequence \( \{t_i\} \) going to infinity such that \( \lim_{i \to \infty} \sup_{x \in X} \rho(x, x t_i) = 0 \). Then for each \( r \) and sufficiently large values \( i \), the diameter of \( K, t_i \) is less than \( \varepsilon \). Therefore for sufficiently large values \( i \) the total change in the angular variable of \( f(x) \) along \( K t_i \) is less than \( n \) radians, where \( n \) is the number of subarcs into which \( K \) was divided. The same statement obviously holds for \( K \) itself.

Now for each \( i \) we will define a map \( \phi_i \) sending the rectangle in the \((U, V)\) plane bounded by the straight lines \( V = 0, V = t_i, U = 0 \) and \( U = 1 \) into \( X \). Any point \((U, 0)\) on the unit interval along the \( U \)-axis is to be sent by \( \phi_i \) into the curve \( K \) in such a way that \( \phi_i \) taken on this interval simply gives the original parametrization of the parametrized curve \( K \). An arbitrary point \((U, V)\) in the rectangle is to be sent by \( \phi_i \) into the point that \( \phi_i(U, 0) \) goes into under the flow after time \( V \). Thus \( \phi_i \) sends vertical lines in the rectangle into orbits in the space \( X \). In particular the segment of the boundary of the rectangle lying on the line \( U = 0 \) gets sent into the orbit going from \( x_1 \) to \( x_1 t_i \), and the corresponding segment on the line \( U = 1 \) goes into the orbit from \( x_2 \) to \( x_2 t_i \). The portion of the boundary along the line \( V = 0 \) goes into \( K \), and the part lying on the line \( V = t_i \) goes into \( K t_i \).

The function \( f(x) \) on \( X \) now gives a function \( f(\phi_i(U, V)) \) on the rectangle. Since the first Betti number of a two-cell is zero, this function on the rectangle has a logarithm, so the total change in the angular variable around the boundary of the rectangle is zero. This can now be applied to curves in \( X \) which are the image of the boundary. Therefore
\[
\Delta K \arg f + \Delta_{(z_2, z_2 t_i)} \arg f - \Delta_{K t_i} \arg f - \Delta_{(x_1, x_1 t_i)} \arg f = 0
\]
or
\[
\Delta_{(z_2, z_2 t_i)} \arg f - \Delta_{(x_1, x_1 t_i)} \arg f = \Delta_{K t_i} \arg f - \Delta K \arg f.
\]

However the right hand side of this last equation is bounded and in fact is smaller than \( 2n \) radians. Therefore, dividing by \( 2\pi t_i \) and letting \( i \) go to infinity,
\[
\lim_{i \to \infty} (1/(2\pi t_i)) \Delta_{(x_1, x_1 t_i)} \arg f = \lim_{i \to \infty} (1/(2\pi t_i)) \Delta_{(x_2, x_2 t_i)} \arg f. \]
an arbitrary function in \( C(X) \) this shows that the asymptotic cycle associated with \( x_1 \) is the same as that associated with \( x_2 \). Thus the flow is spectrally determinate and the asymptotic cycle is defined and has the same value for all orbits, whether quasi-regular or not. In particular then for a recurrent flow any two closed orbits determine elements of the first Betti group which are rationally dependent.

7. Cross sections

Definition. For the purposes of this paper, a subset \( K \) is said to be a cross-section of \( X \) provided \( K \) is closed and the mapping \( \phi \) of \( K \times R \) into \( X \) sending \((p, t)\) into \( pt \) is a local homeomorphism onto \( X \). This is the same as what was called a “surface of section” by G. D. Birkhoff in his book “Dynamical Systems”. Up to the point where asymptotic cycles enter the picture, the discussion given below follows Birkhoff (p. 144).

It is easy to see that if \( K \) is any closed set and the mapping \( \phi \) defined above is onto all of \( X \), then a necessary and sufficient condition that \( K \) be a cross-section is that for some \( \varepsilon > 0 \) \( \phi \) maps the cartesian product of \( K \) with the open interval \((-\varepsilon, \varepsilon)\) homeomorphically onto an open subset of \( X \). Next, suppose that \( K \) is any cross section of \( X \). For any point \( x \in X \), let \( t_x \) be the smallest non-negative real number \( t \) such that there exists a point \( p \) in \( K \) for which \( x = pt \). Since the transformation sending \( x \) into \( xt \) is a homeomorphism there is only one \( p \) for which \( x = pt_x \). Call this point \( p_x \). Next, let \( T_x \) be the smallest positive value of \( T \) for which \( p_x \) \( T \) belongs to \( K \). (Clearly, such a \( T_x \) exists.) We now define \( f_K(X) \) to equal \( (2\pi T_x)^{-1} \) \( f \). It is easy to see that \( f_K(X) \) is in \( C(X) \).

Lemma. Let \( f(x) \) be in \( C(X) \) and suppose \( f(x) \) is differentiable with respect to the flow, and that \((f(x))/(2\pi if(x)) > 0 \) for all \( x \). If we let \( K \) be the set of points for which \( f(x) = 1 \), then \( K \) is a cross-section and \( f_K(x) \) and \( f(x) \) are congruent mod \( R(X) \).

Proof. Let \( \phi \) be the mapping of \( K \times R \) into \( X \) sending \((p, t)\) into \( pt \). Obviously \( \phi \) is a continuous mapping onto all of \( X \). Let \((p, t)\) be a point of \( K \times R \) and suppose \((p_i, T_i)\) and \((q_i, S_i)\) are two sequences of points in \( K \times R \) converging to \((p, t)\). Suppose \( p_i T_i = q_i S_i \). Then \( p_i T_i - S_i = q_i \). Therefore \( f(p_i(T_i - S_i)) = 1 \) so that \((1/(2\pi))\Delta_{p_i, p_i(T_i - S_i)} \ arg f \) is an integer, i.e.

\[
\int_0^{T_i - S_i} \frac{f'(p_i t)}{f(p_i t)} \ dt
\]

is an integer.

Since the integrand is positive and \( T_i - S_i \) goes to zero, for sufficiently large values of \( i \), \( T_i = S_i \) and \( p_i = q_i \). This shows that \( \phi \) is locally one-to-one.

Next let \( O \) be an open subset of \( K \times R \) and suppose \((p, t)\) belongs to \( O \). Let \( \{q_i\} \) converge to \( pt \) in \( X \). Then \( \{f(q_i(-t))\} \) converges to \( f(p) = 1 \). Let \( m = \inf \frac{f'(x)}{2\pi if(x)} \). Then \((1/(2\pi))\Delta_{(x_1, x_2)} \ arg f \ | \geq m \ | t_1 - t_2 | \) for any \( x \in X \) and any \( t_1, t_2 \). From this it follows that it is possible to choose a sequence of real numbers \( t_i \) such that \( t_i \) converges to \(-t \) and \( f(q_i t_i) = 1 \) for all \( i \). There-
fore \( q_{i,t} \) belongs to \( K \) for every integer \( i \). Since \( t_i \) converges to \( -t \), \( q_{i,t} \) converges to \( p \) and for sufficiently large values of \( i \), \( (q_{i,t_i}, -t_i) \) belongs to \( O \). However \( \phi(q_{i,t_i}, -t_i) = q_i \). This shows that \( \phi \) is an open mapping. Since we already know that \( \phi \) is locally one-to-one it follows that \( \phi \) is a local homeomorphism and therefore \( K \) is a cross-section.

Next, let \( \psi \) be the mapping of \([0, 1] \times K \) into \( X \) which sends \((\alpha, p)\) into

\[
p(\alpha T_p).
\]

Since \( T_p \) is a continuous function of \( p \) in \( K \), \( \psi \) is a continuous mapping onto all of \( X \). If we let \( g(\alpha, p) = \exp (2\pi i\alpha) \), then \( g(\alpha, p) = f_K(\psi(\alpha, p)) \). Now it is easy to see that there is one and only one real-valued function \( H(\alpha, p) \) such that

\[
H(0, p) = 0 \quad \text{and} \quad f(\psi(\alpha, p)) = \exp (2\pi iH(\alpha, p)).
\]

Obviously \( H(\alpha, p) \) is continuous and \( H(1, p) = 1 \) for all \( p \) in \( K \). Then

\[
f_K(\psi(\alpha, p))/f(\psi(\alpha, p)) = \exp (2\pi i(\alpha - H(\alpha, p))).
\]

Since \( \alpha - H(\alpha, p) = 0 \) for \( \alpha = 0 \) or \( 1 \), there is a real valued function \( h(x) \) defined on \( X \) for which \( \alpha - H(\alpha, p) = h(\psi(\alpha, p)) \). Obviously \( h(x) \) is continuous and \( f_K(x)/f(x) = \exp (2\pi i h(x)) \). Therefore \( f_K(x) \) is congruent to \( f(x) \) mod \( R(X) \) and the proof of the lemma is completed.

**Theorem.** Let \( f(x) \in C(X) \). A necessary and sufficient condition that there exists a cross-section \( K \) for which \( f_K(x) \) is congruent to \( f(x) \) mod \( R(X) \) is that \( A_\mu(f) > 0 \) for every positive invariant measure \( \mu \).

**Proof.** First suppose that such a cross-section \( K \) exists. If we let \( M = \sup_{p \in K} T_p \) and if \( x \) is any quasi-regular point,

\[
A(x f_K) = \lim_{T \to \infty} (1/(2\pi T)) \Delta(x, x T) \arg f_K \geq 1/M.
\]

Thus for any positive invariant measure \( \mu \), \( A_\mu(f) = A_\mu(f_K) = \int_X A(x f_K) d\mu(x) > 0 \).

Next suppose that \( A_\mu(f) > 0 \) for every positive invariant measure \( \mu \). There is no loss in generality in assuming that \( f \) is differentiable with respect to the flow, since if this were not the case we could replace \( f \) by a function equivalent to \( f \) mod \( R(X) \) and having this property. Now let \( B(X) \) be the space of all continuous real-valued functions \( g(x) \) on \( X \) with \( \| g(x) \| = \sup_{x \in X} |g(x)| \). As in the previous section let \( D \) be the space of functions which are derivatives of real valued functions differentiable with respect to the flow. The statement that \( A_\mu(f) > 0 \) for all positive \( \mu \) says that

\[
\int_X (f'(x))/\exp(2\pi i f(x)) \, d\mu(x) > 0 \quad \text{for all positive } \mu.
\]

If \( D \oplus \{(f'(x))/\exp(2\pi i f(x)) \} \) did not intersect the cone of positive functions in \( B(X) \) it would follow from the Hahn-Banach theorem that for some positive \( \mu \),

\[
\int_X (f'(x))/\exp(2\pi i f(x)) \, d\mu(x) = 0.
\]

Thus there is a function \( H(x) \) which is differentiable with respect to the flow and has the property that \( H'(x) + (f'(x))/\exp(2\pi i f(x)) > 0 \). Let \( \hat{f}(x) = f(x) \exp(2\pi i H(x)) \). Then \( \hat{f}'(x) \) exists and \( \hat{f}'(x)/\exp(2\pi i \hat{f}(x)) \)
is positive, and so by the lemma proved above, if we let \( K \) be the set of points \( x \) for which \( f(x) = 1 \), \( K \) is a cross-section and \( f_\mathcal{K}(x) \) is equivalent to \( f(x) \). Since \( f(x) \) is equivalent to the original \( f(x) \mod R(X) \), the theorem is proved.

Now recall that the conjugate space of \( B(X) \) is the space of all finite real-valued measures \( \mu \) defined on the Borel sets of \( X \). If we put the weak* topology on this space of functionals, it is well known that the subspace consisting of normalized positive measures invariant with respect to the flow is compact, and is in fact the convex closure in the weak* topology of the set of measures \( \mu_p \) arising from quasi-regular points \( p \). Since for any \( f(x) \in C(X) \), the map sending \( \mu \) into \( A_\mu(f) \) is continuous in the weak* topology, the same can be said for the map into the first Betti group taking \( \mu \) into \( A_\mu \). From this it follows that the subset of the first Betti group consisting of all elements \( A_\mu \) is a compact convex set \( \mathfrak{A} \) which is the convex closure of those elements which are of the type \( A_p \) for some quasi-regular point \( p \).

From the previous theorem it now follows that our flow admits of a cross-section if and only if \( \mathfrak{A} \) does not contain the zero element of the first Betti group. From this there follows a practical criterion for proving in certain cases that a cross-section does not exist. Suppose that for some finite set of points \( p_1, \ldots, p_k \) we can find the associated asymptotic cycles \( A_{p_1}, \ldots, A_{p_k} \). (In particular this will be possible whenever we have located some periodic orbits.) The smallest convex set containing \( A_{p_1}, \ldots, A_{p_k} \) is clearly contained in \( \mathfrak{A} \), so if this set contains the origin there can be no cross-section.

Finally, we note that if the flow is spectrally determinate a cross-section exists if and only if the asymptotic cycle of the flow is different from zero. In particular, if the flow is periodic a cross-section exists unless each of the closed orbits is homologous to zero, provided \( X \) is arcwise connected. This is so because a periodic flow is recurrent and we have proved that a recurrent flow is spectrally determinate.

8. Hamiltonian systems\(^1\) [2]

Let \( X \) be a compact differentiable manifold of class \( C^q \) and suppose \( \omega \) is a closed non-singular two form of class \( C^1 \) defined over \( X \). Since \( \omega \) is non-singular the dimension of \( X \) must be even. By definition a canonical system of coordinates is a local coordinatization from a sphere in \( 2n \)-dimensional euclidean space with coordinates \((p_i, q_i)\) such that \( \omega = \sum dp_i \wedge dq_i \). The fact that in general it is possible to cover \( X \) by such canonical coordinate systems follows from a well-known theorem of Pfaff.

Next let \( \alpha \) be a closed one-form of class \( C^1 \) defined over \( X \). In any canonical coordinate system one can find a function \( H(p_i, q_i) \) such that \( \alpha = dh \). Since \( H \) is uniquely determined to within an additive constant, the differential equations

\[
\frac{dq_i}{dt} = \partial H / \partial p_i, \quad \frac{dp_i}{dt} = -\partial H / \partial q_i
\]

\(^1\) Hamiltonian systems of this type have been considered in the large recently by G. Reeb.
are determined by the closed one-form $\alpha$. Moreover from the classical theory of Hamiltonian systems it follows that the differential equations obtained in and two canonical coordinate systems are compatible wherever the coordinatized regions overlap. Thus we obtain from these differential equations a flow on the manifold. We denote the differentiation associated with the flow by $D_\alpha$.

Next, let $\alpha$ and $\beta$ be any two given differentiable closed one-forms. In a canonical coordinate system let $\alpha = dH_1$ and $\beta = dH_2$. The classical Poisson bracket of $H_1$ and $H_2$ is

$$\sum \frac{\partial H_1}{\partial p_i} \frac{\partial H_2}{\partial q_i} - \frac{\partial H_1}{\partial q_i} \frac{\partial H_2}{\partial p_i}.$$  

It is obvious that this expression would be the same for any local integrals $H_1$ and $H_2$ of the one-forms $\alpha$ and $\beta$; moreover it is a standard result that this expression has the same value in any canonical coordinate system. Thus $\alpha$ and $\beta$ determine a function on $X$ by the above formula; we denote this function by $[\alpha, \beta]$ and notice that $[\alpha, \beta] = -[\beta, \alpha]$.

We now introduce the measure $\mu$ determined by the $2n$-form which is obtained by taking the exterior product of $\omega$ with itself $n$ times. We assume that $\omega$ is such that $\mu(X) = 1$. It follows from a well known theorem of Liouville’s that for any closed one-form $\alpha$, the flow we have associated with $\alpha$ by means of Hamilton’s equation has $\mu$ as an invariant measure. Notice next that if $\beta = dH_2$, where $H_2$ is a function defined over the entire manifold $X$, then $[\alpha, \beta] = D_\alpha H_2$.

Since $\mu$ is an invariant measure, $\int_X [\alpha, dH_2(x)] d\mu(x) = \int_X D_\alpha (H_2(x)) d\mu(x) = 0$. By the skew-symmetry of the Poisson bracket it follows that if $\alpha = dH_1(x)$ and $\beta$ is any closed one-form, $\int_X [dH_1, \beta] d\mu(x) = 0$. By the bilinearity of the Poisson bracket it follows that $\int_X [\alpha, \beta] d\mu(x)$ is completely determined by the cohomology classes to which $\alpha$ and $\beta$ belong.

Next let $f(x) \in C(X)$ and suppose $f(x)$ is of class $C^2$. We know that there is such an $f(x)$ in every equivalence class mod $R(X)$. It was pointed out in Section three that if we represent $f(x)$ locally as $\exp(2\pi i H_2(x))$ then $dH_2$ gives a unique closed one-form $\beta_f$ defined over all of $X$, and the map sending $f$ into $\beta_f$ induces the natural mapping of $C(X)/R(X)$ into the quotient of the space of closed one-forms modulo the bounding one-forms. Now let $\alpha$ be a fixed one-form and $f$ a fixed element of $C(X)$. From now on differentiation will mean differentiation with respect to the flow determined by $\alpha$. It is clear that since locally $f(x) = \exp(2\pi i H_2(x))$, $(1/(2\pi i))(f'(x))/(f(x)) = H_2(x) = D_\alpha H_2(x) = [\alpha, \beta_f]$. Thus

$$1/(2\pi i) \int_X (f'(x))/(f(x)) d\mu(x) = \int_X [\alpha, \beta_f] d\mu.$$  

The fact that the integral of this Poisson bracket depends only on the cohomology class of $\beta_f$ is in accordance with the theory developed previously; it simply re-
minds us that \( 1/(2\pi i) \int f'(x)/f(x) \, d\mu(x) \) depends only on the equivalence
class to which \( f(x) \) belongs mod \( R(X) \). The fact that this integral depends only
on the cohomology class to which \( \alpha \) belongs, together with the fact that the
Poisson bracket is bilinear yields something more interesting, however.

**Theorem.** The \( \mu \)-asymptotic cycle \( A_\mu \) associated with the flow obtained from
a closed one-form \( \alpha \) is completely determined by the cohomology class to which \( \alpha \) be-
longs. Moreover, the mapping which sends cohomology classes into the associated
\( \mu \)-asymptotic cycles is a linear mapping of the first cohomology group into the first
homology group.

As a special case we consider Hamiltonian flows on even-dimensional multi-
tori. Let \( H(p_i, q_i) \) be a function of class \( C^2 \) in euclidean \( 2n \)-space and suppose
that in the Hamiltonian equations

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},
\]

the functions on the right hand side of these equations are periodic of period one
with respect to each of the variables \( p_i, q_i \). Then there exists a function \( K(p_i, q_i) \)
which is also of period one with respect to each of the variables, together with
constants \( a_i, b_i \) such that

\[
H(p_i, q_i) = K(p_i, q_i) + \sum a_ip_i + b_iq_i. \]

If we consider these equations as inducing a flow on the multi-torus, and let \( \mu \) be
ordinary euclidean measure on this multi-torus, the theorem proved above shows that the
\( (\mu) \)-asymptotic cycle for the Hamiltonian \( H(p_i, q_i) \) is the same as the \( (\mu) \)-
asymptotic cycle we would get using \( \sum a_ip_i + b_iq_i \) as our Hamiltonian. How-
ever, the equations of motion for this Hamiltonian are

\[
\frac{dq_2}{dt} = a_i, \quad \frac{dp_2}{dt} = -b_i;
\]

and it is obvious from geometric considerations that the \( (\mu) \)-asymptotic cycle
is \( \sum (a_iq_i - b_ip_i) \); where \( g_{q_i} \) and \( g_{p_i} \) are the fundamental cycles associated
with the variables \( q_i \) and \( p_i \).

As a final remark, we note that in the case of Hamiltonian flows on the two
dimensional torus, this tells us immediately what the winding number of Poincaré
are. Also, even on the multi-torus, we can apply the results of the previous section
on continuous eigenfunctions to see which eigenvalues are possible.

**Bibliography**

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