SRB measures without symbolic dynamics or dominated splittings

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Joint work with Dmitry Dolgopyat and Yakov Pesin
1. Introduction and classical results
   - Definition of SRB measure
   - Examples, known and otherwise

2. General method
   - Decomposing the space of invariant measures
   - Recurrence to compact sets

3. Recurrence to $S_n(K)$
   - Sequences of local diffeomorphisms
   - Frequency of large admissible manifolds
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   - Maps on the boundary of Axiom A: Slowdown, no shear
   - Maps on the boundary of Axiom A: Slowdown and shear
Physically meaningful invariant measures

- $M$ a compact Riemannian manifold
- $f : M \to M$ a $C^{1+\varepsilon}$ diffeomorphism
- $\mathcal{M}$ the space of Borel measures on $M$
- $\mathcal{M}(f) = \{\mu \in \mathcal{M} \mid \mu$ is $f$-invariant$\}$
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**Birkhoff ergodic theorem.** If $\mu \in \mathcal{M}(f)$ is ergodic then it describes the statistics of $\mu$-a.e. trajectory of $f$: for every integrable $\varphi$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu$$
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To be “physically meaningful”, a measure should describe the statistics of \textit{Lebesgue}-a.e. trajectory.
SRB measures

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SRB measures

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- ...many systems are not conservative.
- Interesting dynamics often happen on a set of Lebesgue measure zero.
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...many systems are not conservative.

Interesting dynamics often happen on a set of Lebesgue measure zero.

“absolutely continuous” $\Rightarrow$ “a.c. on unstable manifolds”

$\mu \in \mathcal{M}(f)$ is an SRB measure if

1. all Lyapunov exponents non-zero;
2. $\mu$ has a.c. conditional measures on unstable manifolds.

Ergodic SRB measures are physically meaningful.

Goal: Prove existence of an SRB measure.
SRB measures are known to exist in the following settings.

- Uniformly hyperbolic $f$ (Sinai, Ruelle, Bowen)
- Partially hyperbolic $f$ with positive/negative central exponents (Alves–Bonatti–Viana, Burns–Dolgopyat–Pesin–Pollicott)

Key tool is a dominated splitting $T_x M = E^s(x) \oplus E^u(x)$.

1. $E^s, E^u$ depend continuously on $x$.
2. $\angle(E^s, E^u)$ is bounded away from 0.

Both conditions fail for non-uniformly hyperbolic $f$. 
Non-uniformly hyperbolic maps

The Hénon maps $f_{a,b}(x, y) = (a - x^2 - by, x)$ are a perturbation of the family of logistic maps $g_a(x) = a - x^2$.

1. $g_a$ has an absolutely continuous invariant measure for “many” values of $a$. (Jakobson)
2. For $b$ small, $f_{a,b}$ has an SRB measure for “many” values of $a$. (Benedicks–Carleson, Benedicks–Young)
3. Similar results for “rank one attractors” – small perturbations of one-dimensional maps with non-recurrent critical points. (Wang–Young)

Genuine non-uniform hyperbolicity, but only one unstable direction, and stable direction must be strongly contracting.
Other non-uniformly hyperbolic maps

Other examples:

1. Hénon $f_{a,b}(x, y) = (a - x^2 - by, x)$ for $b \gg 0$.
2. Generalised Hénon $f_{a,b}(x, y, z) = (a - y^2 - bz, x, y)$: expect to have two unstable directions, so not rank one.
3. Large perturbations of Axiom A maps: Katok construction (slowdown near hyperbolic fixed point), no dominated splitting; slowdown + shear, no continuous splitting.
4. Small perturbations of maps with SRB measures: either local or global.

Goal: Develop a method for establishing the existence of an SRB measure that can be applied to these and other examples.
Constructing invariant measures

- $f$ acts on $\mathcal{M}$ by $f_* : m \mapsto m \circ f^{-1}$.
- Fixed points of $f_*$ are invariant measures.
- Césaro averages + weak* compactness $\Rightarrow$ invariant measures:
  \[ \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m \]
  \[ \mu_{n_j} \rightarrow \mu \in \mathcal{M}(f) \]
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$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f^k_* m$$

$$\mu_{n_j} \to \mu \in \mathcal{M}(f)$$

Idea: $m = \text{volume} \Rightarrow \mu$ is an SRB measure.

$H = \{x \in M \mid \text{all Lyapunov exponents non-zero at } x\}$

$S = \{\nu \in \mathcal{M} \mid \nu(H) = 1, \nu \text{ a.c. on unstable manifolds}\}$

- $S \cap \mathcal{M}(f) = \{\text{SRB measures}\}$
- $S$ is $f_*$-invariant, so $m \in S \Rightarrow \mu_n \in S$ for all $n$.
- $S$ is not compact. So why should $\mu$ be in $S$?
Non-uniform hyperbolicity in $\mathcal{M}$

Theme in NUH: choose between invariance and compactness.

Replace unstable manifolds with $n$-admissible manifolds $V$.

$$d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\lambda k}d(x, y) \text{ for all } 0 \leq k \leq n \text{ and } x, y \in V$$

$$S_n = \{\nu \text{ supp. on and a.c. on } n\text{-admissible manifolds, } \nu(H) = 1\}$$
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This set of measures has various non-uniformities.

1. Value of $C, \lambda$ in definition of $n$-admissibility.
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1. Value of $C, \lambda$ in definition of $n$-admissibility.
2. Size and curvature of admissible manifolds.
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2. Size and curvature of admissible manifolds.
3. $\|\rho\|$, where $\rho$ is density wrt. leaf volume.
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This set of measures has various non-uniformities.

1. Value of $C, \lambda$ in definition of $n$-admissibility.
2. Size and curvature of admissible manifolds.
3. $\| \rho \|$, where $\rho$ is density wrt. leaf volume.

Given $K > 0$, let $S_n(K)$ be the set of measures for which these non-uniformities are all controlled by $K$.

large $K \implies$ worse non-uniformity

$S_n(K)$ is compact, but not $f_*$-invariant.
Decomposing the space of invariant measures

Non-uniformities controlled by $K$

Admissible manifold $V$ near $x$ defined by

- decomposition $T_x M = G \oplus F$ with $\alpha = \angle (G, F)$,
- $C^{1+\varepsilon}$ function $\psi : G \cap B(0, r) \to F$ with $\|D\psi\| \leq \gamma$ and $|D\psi|_{\varepsilon} \leq \kappa$ such that $V = \exp_x(\text{graph } \psi)$.

Density $\rho \in C^\varepsilon(V)$ and backwards dynamics satisfy

- $L^{-1} \leq \rho(x) \leq L$ and $\|\rho\|_{C^\varepsilon} \leq L$,
- $d(f^{-k}(x), f^{-k}(y)) \leq C e^{-\lambda k} d(x, y)$.

$K$ controls all the quantities $\alpha, r, \gamma, \kappa$ (geometry of the admissible manifold), $L$ (density function), and $C, \lambda$ (dynamics).
Introduction

General method

Recurrence to compact sets

Applications

Recurrence to compact sets

Conditions for existence of an SRB measure

- $M$ be a compact Riemannian manifold, $U \subset M$ open, $f: U \to M$ a local diffeomorphism with $f(U) \subset U$.

- Let $\mu_n$ be a sequence of measures whose limit measures are all invariant. (In applications, $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f^k \star \text{Leb}$.)

- Fix $K > 0$, write $\mu_n = \nu_n + \zeta_n$, where $\nu_n \in S_n(K)$.

Theorem (C.–Dolgopyat–Pesin 2011)

If $\mu_{n_k} \to \mu$ and $\lim_{n_k \to \infty} \|\nu_{n_k}\| > 0$ and , then some ergodic component of $\mu$ is an SRB measure for $f$. 
Recurrence to compact sets

Conditions for existence of an SRB measure

- $M$ be a compact Riemannian manifold, $U \subset M$ open,\n  \( f : U \to M \) a local diffeomorphism with $f(U) \subset U$.

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The question now becomes: How do we obtain recurrence to the\nset $S_n(K)$?
We use local coordinates to write the map $f$ along a trajectory as a sequence of local diffeomorphisms.

- $\{f^n(x) \mid n \geq 0\}$ is a trajectory of $f$
- $U_n \subset T_{f^n(x)}M$ is a small neighbourhood of 0
- $f_n : U_n \to \mathbb{R}^d = T_{f^{n+1}(x)}M$ is the map $f$ in local coordinates
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- $U_n \subset T_{f^n(x)}M$ is a small neighbourhood of 0
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Decompose $\mathbb{R}^d = T_xM = E^u_0 \oplus E^s_0$, let $E^{u,s}_{n+1} = Df_n(E^{u,s}_n)$.

- Want $E^u_n$ and $E^s_n$ asymptotically expanding and contracting.
- Want $\lim_n \angle(E^u_n, E^s_n) > 0$.  
- ($\lim_n \angle(E^u_n, E^s_n) > 0 \text{ is probably unavoidable.}$)
Controlling hyperbolicity and regularity

\[ \mathbb{R}^d = T_{f^n(x)} M = E_n^u \oplus E_n^s \]

\[ f_n = (A_n \oplus B_n) + s_n \]

Start with an admissible manifold \( V_0 \) tangent to \( E_0^u \) at 0 and push it forward: \( V_{n+1} = f_n(V_n) \).
Controlling hyperbolicity and regularity

\[ \mathbb{R}^d = T_{f^n(x)}M = E_n^u \oplus E_n^s \quad f_n = (A_n \oplus B_n) + s_n \]

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\[ V_n = \text{graph } \psi_n = \{ v + \psi_n(v) \} \quad \psi_n : B(E_n^u, r_n) \rightarrow E_n^s \]

Need to control the size \( r_n \) and the regularity \( \|D\psi_n\|, \|D\psi_n\|_\varepsilon \).
Controlling hyperbolicity and regularity

Consider the following quantities:

\[
\lambda_n^u = \log(\|A_n^{-1}\|^{-1})
\]
\[
\lambda_n^s = \log \|B_n\|
\]
\[
\alpha_n = \angle(E_n^u, E_n^s)
\]
\[
C_n = |D\psi_n|_\varepsilon
\]

\[
V_n = \text{graph } \psi_n = \{ v + \psi_n(v) \}
\]
\[
\psi_n : B(E_n^u, r_n) \to E_n^s
\]

Need to control the size \(r_n\) and the regularity \(\|D\psi_n\|, |D\psi_n|_\varepsilon\).
Classical Hadamard–Perron results

Uniform case: Constants such that

- \( \lambda_n^s \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda_n^u \)
- \( \alpha_n \geq \bar{\alpha} > 0 \)
- \( C_n \leq \bar{C} < \infty \)

Then \( V_n \) has uniformly large size: \( r_n \geq \bar{r} > 0 \).
**Classical Hadamard–Perron results**

**Uniform case:** Constants such that
- $\lambda_n^s \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda_n^u$
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Then $V_n$ has uniformly large size: $r_n \geq \bar{r} > 0$.

**Non-uniform case:** $\lambda_n^s, \lambda_n^u, \alpha_n$ still uniform, but $C_n$ not.

$C_n$ grows slowly $\Rightarrow$ $r_n$ decays slowly
Sequences of local diffeomorphisms

Classical Hadamard–Perron results

Uniform case: Constants such that

- $\lambda_s^n \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda_u^n$
- $\alpha_n \geq \bar{\alpha} > 0$
- $C_n \leq \bar{C} < \infty$

Then $V_n$ has uniformly large size: $r_n \geq \bar{r} > 0$.

Non-uniform case: $\lambda_s^n, \lambda_u^n, \alpha_n$ still uniform, but $C_n$ not.

- $C_n$ grows slowly $\Rightarrow r_n$ decays slowly

We want to consider the case where

- $\lambda_s^n < 0 < \lambda_u^n$ may fail (may even have $\lambda_u^n < \lambda_s^n$)
- $\alpha_n$ may become arbitrarily small
- $C_n$ may become arbitrarily large (no control on speed)
Usable hyperbolicity

In order to define $\psi_{n+1}$ implicitly, we need control of the regularity of $\psi_n$. Control $||D\psi_n||$ and $|D\psi_n|_\varepsilon$ by decreasing $r_n$ if necessary. So how do we guarantee that $r_n$ becomes “large” again?
Usable hyperbolicity

In order to define $\psi_{n+1}$ implicitly, we need control of the regularity of $\psi_n$. Control $\|D\psi_n\|$ and $|D\psi_n|_{\varepsilon}$ by decreasing $r_n$ if necessary. So how do we guarantee that $r_n$ becomes “large” again?

**Defect** – splitting not dominated: $d_n = \max \left(0, \frac{1}{\varepsilon} (\lambda_n^s - \lambda_n^u)\right)$

**Distortion** – large nonlinearity, small angle: $\beta_n = C_n (\sin \alpha_{n+1})^{-1}$

Fix a threshold value $\bar{\beta}$ and define the usable hyperbolicity:

$$\lambda_n = \begin{cases} 
\lambda_n^u - d_n & \text{if } \beta_n \leq \bar{\beta}, \\
\min \left(\lambda_n^u - d_n, \frac{1}{\varepsilon} \log \frac{\beta_{n-1}}{\beta_n} \right) & \text{if } \beta_n > \bar{\beta}.
\end{cases}$$

Continuous dominated splitting $\Rightarrow \lambda_n = \lambda_n^u$
Key criterion will be positive usable hyperbolicity:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k > 0$$

for some $\bar{\beta}$

One way to establish this is to have both of the following:

1. Expansion beats defect:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^u_k - d_k > 0$$

2. Distortion is almost bounded: Let $\Gamma^{\bar{\beta}} = \{ n \mid \beta_n > \bar{\beta} \}$. Then $\Gamma^{\bar{\beta}}$ has arbitrarily small upper asymptotic density.
A Hadamard–Perron theorem

- $F_n = f_{n-1} \circ \cdots \circ f_1 \circ f_0 : U_0 \to \mathbb{R}^d = T_{f_n(x)}M$
- $V_0 \subset \mathbb{R}^d$ a $C^{1+\varepsilon}$ manifold tangent to $E^u_0$ at 0
- $V_n(r) =$ connected component of $F_n(V_0) \cap B(r)$ containing 0

**Theorem (C.–Dolgopyat–Pesin 2011)**

Suppose $\lim \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k > \bar{\chi} > 0$ for some $\bar{\beta}$. Then there exist constants $\bar{\alpha}, \bar{\gamma}, \bar{\kappa}, \bar{r} > 0$ and a set $\Gamma \subset \mathbb{N}$ with positive lower asymptotic frequency such that for every $n \in \Gamma$,

1. $\angle(E^u_n, E^s_n) \geq \bar{\alpha}$;
2. $V_n(\bar{r}) = \text{graph } \psi_n$ and $\|D\psi_n\| \leq \bar{\gamma}$, $|D\psi_n|_\varepsilon \leq \bar{\kappa}$;
3. if $F_n(x), F_n(y) \in V_n(\bar{r})$, then for every $0 \leq k \leq n$,

$$\|F_n(x) - F_n(y)\| \geq e^{(n-k)\bar{\chi}} \|F_k(x) - F_k(y)\|.$$
Idea of proof

Start with $V_0$, study $V_n = F_n(V_0)$. Choose $r_n, \gamma_n, \kappa_n$ such that

- $V_n(r_n) = \text{graph } \psi_n$
- $\|D\psi_n\| \leq \gamma_n$ and $|D\psi_n|_\epsilon \leq \kappa_n$.

Can improve $\gamma_n, \kappa_n$ at the cost of reducing $r_n$, or vice versa. Give conditions on “goodness parameters” $r_n, \gamma_n, \kappa_n$; inequalities in terms of $\lambda_n^u, \lambda_n^s, \beta_n$. 

Frequency of large admissible manifolds
Idea of proof

Start with $V_0$, study $V_n = F_n(V_0)$. Choose $r_n, \gamma_n, \kappa_n$ such that

- $V_n(r_n) = \text{graph} \, \psi_n$
- $\|D\psi_n\| \leq \gamma_n$ and $|D\psi_n|_\varepsilon \leq \kappa_n$.

Can improve $\gamma_n, \kappa_n$ at the cost of reducing $r_n$, or vice versa. Give conditions on “goodness parameters” $r_n, \gamma_n, \kappa_n$; inequalities in terms of $\lambda_n^u, \lambda_n^s, \beta_n$.

Truncate parameters at threshold values $\bar{r}, \bar{\gamma}, \bar{\kappa}$:

- define goodness $g_n$ by $g_0 = 1$ and $g_{n+1} = \min(1, e^{\lambda_n} g_n)$;
- $r_n = \bar{r} g_n$, $\gamma_n = \bar{\gamma}$, $\kappa_n = \bar{\kappa} g_n^{-\varepsilon}$. 
Idea of proof

Start with $V_0$, study $V_n = F_n(V_0)$. Choose $r_n, \gamma_n, \kappa_n$ such that
- $V_n(r_n) = \text{graph} \psi_n$
- $\|D\psi_n\| \leq \gamma_n$ and $|D\psi_n|\varepsilon \leq \kappa_n$.

Can improve $\gamma_n, \kappa_n$ at the cost of reducing $r_n$, or vice versa. Give conditions on “goodness parameters” $r_n, \gamma_n, \kappa_n$; inequalities in terms of $\lambda_n^u, \lambda_n^s$, and $\beta_n$.

Truncate parameters at threshold values $\bar{r}, \bar{\gamma}, \bar{\kappa}$:
- define goodness $g_n$ by $g_0 = 1$ and $g_{n+1} = \min(1, e^{\lambda_n} g_n)$;
- $r_n = \bar{r} g_n, \gamma_n = \bar{\gamma}, \kappa_n = \bar{\kappa} g_n^{-\varepsilon}$.

positive asymptotic rate of usable hyperbolicity
$\Rightarrow$ positive frequency of usable hyperbolic times (Pliss’ lemma)
$\Rightarrow$ thresholded parameters spend enough time at threshold
Key consequence

- $\mu_0 = \text{Leb} \mid \mathcal{V}_0$
- $\mu_n = (f_n^* \mu_0) \mid \mathcal{V}_n(r_n)$ (normalised)
- $\mu_n \in S_n(K)$ for $n \in \Gamma$
- $\nu_N = \frac{1}{N} \sum_{k=0}^{N-1} \mu_n$
- $\nu_N$ has uniformly positive projection to $S_n(K)$ for $N \gg n$
Key consequence

- \( \mu_0 = \text{Leb}_\mid V_0 \)
- \( \mu_n = (f^*_n \mu_0)\mid V_n(r_n) \) (normalised)
- \( \mu_n \in S_n(K) \) for \( n \in \Gamma \)
- \( \nu_N = \frac{1}{N} \sum_{k=0}^{N-1} \mu_n \)
- \( \nu_N \) has uniformly positive projection to \( S_n(K) \) for \( N \gg n \)

Problem: \( \lim \nu_N \) is not invariant because of normalisation.

Key step for applications: Show that the set of points with positive rate of usable hyperbolicity has positive Lebesgue measure. (Either on \( M \) or on \( V_0 \).)
Cone families

Return to a local diffeomorphism $f : U \to M$. Given $x \in M$, a subspace $E \subset T_x M$, and an angle $\theta$, we have a cone

$$K(x, E, \theta) = \{ v \in T_x M \mid \angle(v, E) < \theta \}.$$

$E, \theta$ depend measurably on $x \sim \Rightarrow$ measurable cone family.
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$E, \theta$ depend measurably on $x \rightsquigarrow$ measurable cone family.

Suppose $\exists$ two measurable cone families $K^s(x), K^u(x)$ s.t.

1. $Df(K^u(x)) \subset K^u(f(x))$ for all $x \in A$
2. $Df^{-1}(K^s(f(x))) \subset K^s(x)$ for all $x \in f(A)$
3. $T_xM = E^s(x) \oplus E^u(x)$
Usable hyperbolicity (again)

Measurable transverse cone families $K^s(x), K^u(x) \subset T_x M$.

$$\lambda^u(x) = \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$

$$\lambda^s(x) = \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}.$$

Let $\alpha(x) = \angle(K^s(x), K^u(x))$. Fix $\bar{\alpha} > 0$ and consider

$$d(x) = \max\left(0, \frac{1}{\varepsilon}(\lambda^s(x) - \lambda^u(x))\right),$$

$$\lambda(x) = \begin{cases} 
\lambda^u(x) - d(x) & \text{if } \alpha(x) \geq \bar{\alpha}, \\
\min\left(\lambda^u(x) - d(x), \frac{1}{\varepsilon} \log \frac{\alpha(x)}{\alpha(f^{-1}(x))}\right) & \text{if } \alpha(x) < \bar{\alpha}.
\end{cases}$$
An existence result

Consider points with positive asymptotic usable hyperbolicity:

\[ S^\alpha = \left\{ x \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\} \]
Consider points with positive asymptotic usable hyperbolicity:

$$S^{\tilde{\alpha}} = \left\{ x \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\}$$

**Theorem (C.–Dolgopyat–Pesin 2011)**

*If $\exists \tilde{\alpha} > 0$ such that $\text{Leb } S^{\tilde{\alpha}} > 0$, then $f$ has an SRB measure.*
An existence result

Consider points with positive asymptotic usable hyperbolicity:

\[
S^{-\alpha} = \left\{ x \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\}
\]

**Theorem (C.–Dolgopyat–Pesin 2011)**

If \( \exists \, \bar{\alpha} > 0 \) such that \( \text{Leb } S^{-\bar{\alpha}} > 0 \), then \( f \) has an SRB measure.

**Theorem (C.–Dolgopyat–Pesin 2011)**

Let \( V \) be tangent to \( K^u(x) \) at \( x \). Suppose \( \exists \, \bar{\alpha} > 0 \) such that

\[
\lim_{r \to 0} \frac{m_V(S^{-\bar{\alpha}} \cap B(x, r))}{m_V(B(x, r))} > 0.
\]

Then \( f \) has an SRB measure.
Large perturbations: an indifferent fixed point

\( f \) an Axiom A diffeomorphism, \( f(p) = p \).
- \( f \) has an SRB measure.
- Small perturbations of \( f \) are Axiom A.
- Consider perturbation on boundary of “small”.

Maps on the boundary of Axiom A: Slowdown, no shear
Large perturbations: an indifferent fixed point

$f$ an Axiom A diffeomorphism, $f(p) = p$.

- $f$ has an SRB measure.
- Small perturbations of $f$ are Axiom A.
- Consider perturbation on boundary of “small”.

Near $p$, this is time-1 map of $\dot{x} = Ax$. Fix $\psi : [0, 1] \rightarrow [0, 1]$ s.t.

- $\psi$ is $C^\infty$ on $(0, 1)$;
- $\psi(0) = 0; \psi' > 0$ on $(0, r_0)$; $\psi \equiv 1$ on $[r_0, 1]$;
- $\psi(r) \approx r^\alpha$ near $0$, for some $\frac{1}{2} < \alpha < 1$.

Near $p$, let $g =$ time-1 map for $\dot{x} = \psi(\|x\|^2)Ax$, with $g = f$ outside of $V = B(p, r_0)$.

**Theorem (C.–Dolgopyat–Pesin 2011)**

$g$ has an SRB measure.
Usable hyperbolicity for $g$

- If $f$ has a smooth invariant measure $\mu$, then $\psi(\|x\|^2)^{-1}d\mu$ defines a smooth invariant measure for $g$.
- If the SRB measure for $f$ is not smooth, then the attractor for $f$ is not $g$-invariant.

$f$ is Axiom A $\Rightarrow$ $f$ has invariant cone families $K^u(x)$ and $K^s(x)$
Usable hyperbolicity for \( g \)

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\( f \) is Axiom A \( \Rightarrow \) \( f \) has invariant cone families \( K^u(x) \) and \( K^s(x) \)

- \( K^u(x) \) and \( K^s(x) \) are \( g \)-invariant.
- \( \lambda^u(x) \geq 0 \geq \lambda^s(x) \) and \( \alpha(x) \gg 0 \) for every \( x \).
- \( \lambda(x) = \lambda^u(x) \geq \chi > 0 \) for every \( x \notin V \).

\[
\frac{1}{n} \sum_{k=0}^{n-1} \lambda(g^k(x)) \geq \chi \cdot \frac{1}{n} \#\{0 \leq k < n \mid g^k(x) \notin V\}
\]
Average sojourn times

- \( \tau(x) = \min \{ t \mid g^t(x) \notin V \} \)
- \( G(x) = g^{\tau(x)}(x) \)
- \( \tau_n(x) = \tau(G^{n-1}(x)) \)

Claim: \( \exists R > 0 \) such that \( \lim \frac{1}{n} \sum_{k=1}^{n} \tau_k(x) \leq R \) for Leb-a.e. \( x \).
Maps on the boundary of Axiom A: Slowdown, no shear

**Average sojourn times**

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- \( \Omega(t_1, \ldots, t_n) = \{ x \mid \tau_k(x) = t_k \text{ for } 1 \leq k \leq n \} \)
- \( \text{Leb } \Omega(t) \leq C^n \prod_{k=1}^{n} t_k^{-\gamma} \text{ with } \gamma > 2 \)
- Model \( (\tau_k) \) with i.i.d. \( (T_k) \) such that \( P(T_k = t) = Ct^{-\gamma} \)
- Claim holds using fact that \( E(T_k) < \infty \)
An indifferent fixed point with a shear

Slow down Axiom A $f$ near $p = f(p)$ as before.
An indifferent fixed point with a shear

Slow down Axiom A $f$ near $p = f(p)$ as before.

Let $N: \mathbb{R}^d \to \mathbb{R}^d$ be linear such that

- $N(\mathbb{R}^d) \subset \{0\} \times \mathbb{R}^u \subset \ker N$,

and $\xi: [0, 1] \to [0, 1]$ such that

- $\xi$ is $C^\infty$ on $(0, 1)$;
- $\xi(0) = 1$; $\xi \equiv 0$ on $[r_0, 1]$.

Near $p$, let $g = \text{time-1 map for } \dot{x} = (\psi(\|x\|^2)A + \xi(\|x\|^2)N)x$, with $g = f$ outside of $V = B(p, r_0)$.

**Theorem (C.–Dolgopyat–Pesin 2011)**

$g$ has an SRB measure.
Stable cones for $g$

Shear $\Rightarrow$ stable cone for $f$ is no longer $g$-invariant. Need to

1. establish existence of stable invariant cones $K^s(x)$ for $g$;
2. estimate $\alpha(x) = \angle(K^s(x), K^u(x))$. 
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Claim: This boils down to estimating average sojourn times.

- $A = V \setminus g(V)$ (just entered neighbourhood of $p$)
- $B = g(V) \setminus V$ (just left the neighbourhood of $p$)
- Let $G: A \to B$ and $F: B \to A$ be the induced maps

Need to understand action of $DG$ and $DF$ on the space of $s$-dimensional subspaces of $\mathbb{R}^d$ transverse to $\mathbb{R}^u \times \{0\}$. 
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Need to understand action of $DG$ and $DF$ on the space of $s$-dimensional subspaces of $\mathbb{R}^d$ transverse to $\mathbb{R}^u \times \{0\}$.

- Identify this space with $(\mathbb{R}^u)^s$
- $DG$ acts as a translation (parabolically)
- $DF$ acts as multiplication (hyperbolically)
Maps on the boundary of Axiom A: Slowdown and shear

**Stable cones for $g$ (ctd.)**

\[
\{ E \subset \mathbb{R}^d \mid E \text{ transverse to } \mathbb{R}^u \times \{0\} \} \leftrightarrow (\mathbb{R}^u)^s
\]

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E \rightarrow \mathbb{R}^u \times \{0\} \leftrightarrow \vec{v} \rightarrow \infty
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**Goal:** $\vec{v}$ such that

\[ \vec{v}, \ DG(\vec{v}), \ DF \circ DG(\vec{v}), \ DG \circ DF \circ DG(\vec{v}), \ldots \]

does not go to $\infty$. This corresponds to $E \subset \mathbb{R}^d$ such that

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\]

does not go to $\mathbb{R}^u \times \{0\}$. Given $\vec{v} = (v_1, \ldots, v_s) \in (\mathbb{R}^u)^s$, we have

- $\|DG_x(\vec{v})_j\| \geq \|v_j\| - C\tau(x)$,
- $\|DF_x(\vec{v})_j\| \geq \lambda \|v_j\|$, where $\lambda > 1$. 
Usable hyperbolicity

\[ R_n(x) := \sum_{k=0}^{\infty} C \lambda^{-k} \tau_{n+k+1}(x) \]  

- \((DF \circ DG)B(R_n(x)) \supset B(R_{n+1}(F \circ G(x)))\)
- \(B(R_n(x))\) contains some \(\vec{v}\) whose iterates do not go to \(\infty\)
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bounded average sojourn time

\[ \Rightarrow \text{positive asymptotic rate of usable hyperbolicity} \]

\[ \Rightarrow g \text{ has an SRB measure} \]
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