Randomness and determinism in dynamical systems

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The talk in one slide

**PHENOMENON**

Deterministic systems can exhibit stochastic behaviour over long time scales

**KNOWN**

Mechanism driving this is phase space expansion

**EXAMPLES**

Lorenz equations, expanding maps, logistic map

**RESEARCH**

What happens when expansion is non-uniform?
Predictions in dynamical systems

Key objects:
- $X = \text{phase space for a dynamical system}$.
  
  Points in $X$ correspond to configurations of the system.

- $f : X \rightarrow X$ describes evolution of the state of the system over a single time step. Can also consider continuous-time systems.

Standing assumptions:
- $X \subset \mathbb{R}^n$
- $f$ is continuous
Predictions in dynamical systems

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- $X$ = phase space for a dynamical system. 
  \textit{Points in $X$ correspond to configurations of the system.}
- $f : X \to X$ describes evolution of the state of the system over a single 
time step. \textit{Can also consider continuous-time systems.}

Standing assumptions:
- $X \subset \mathbb{R}^n$
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Predictions rely on finding $f^n(x)$ given $x$.

\textit{initial error} $\Rightarrow$ must compare $f^n(x)$ and $f^n(y)$ when $x \sim y$

\textit{Distinct problem from accounting for discrepancy between model and 
real-world system, or for numerical error in computation of $f^n(x)$.
A mechanism for stochastic behaviour

Fix \( x \sim y \). Two extremes:

- **Stable behaviour:** \( d(f^n x, f^n y) \to 0 \)
  
  *Even better: there is \( p = f(p) \) such that \( f^n x \to p \) for all \( x \)*

- **Unstable behaviour:** \( d(f^n x, f^n y) \) grows quickly

In “chaotic” systems, unstable behaviour is prevalent:

- initial error grows exponentially fast
- prediction \( f^n(x) \) quickly diverges from reality

Another perspective: \( U \subset X \) a small neighbourhood, consider \( f^n(U) \).

- In chaotic systems, diameter of iterates \( f^n(U) \) becomes large (exponentially quickly) no matter how small \( U \) is.
Lorenz equations (1963) – atmospheric dynamics

\[
\begin{align*}
\dot{x} &= \sigma (y - x) & \sigma &= 10 \\
\dot{y} &= x (\rho - z) - y & \rho &= 28 \\
\dot{z} &= xy - \beta z & \beta &= 8/3
\end{align*}
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**Doubling map** $f : S^1 \subset \mathbb{C}$, $z = e^{ix} \mapsto z^2 = e^{i(2x)}$

**Full shift** $\Sigma_2^+ = \{0, 1\}^\mathbb{N}$, $f = \sigma : x_0x_1x_2 \ldots \mapsto x_1x_2x_3 \ldots$. 

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Chaos

Examples

Doubling map

Logistic map

Bifurcation diagram

Summary

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Full shift \( \Sigma^+_2 = \{0, 1\}^\mathbb{N}, f = \sigma : x_0 x_1 x_2 \ldots \mapsto x_1 x_2 x_3 \ldots \)

Logistic map \( f_\lambda : [0, 1] \to [0, 1], x \mapsto \lambda x(1 - x), \lambda \in [0, 4] \)

Code trajectories with 0s and 1s, but don’t get full shift.
Predictions for the doubling map

**Doubling map** \( f: S^1 \circlearrowleft, \ S^1 \subset \mathbb{C}, \ z = e^{ix} \mapsto z^2 = e^{i(2x)} \)

**Full shift** \( \Sigma_2^+ = \{0, 1\}^\mathbb{N}, \ f = \sigma: \ x_0x_1x_2 \ldots \mapsto x_1x_2x_3 \ldots \)

**Predictions are impossible:** If initial error is \( \epsilon \) then error at time \( n \) is \( \epsilon 2^n \).

- Lengthening prediction by time 1 requires doubling initial accuracy.
Chaos
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Predictions are easy: Lebesgue measure $\nu$ on the circle is $f$-invariant

$$\nu(f^{-1}E) = \nu\{z \mid f(z) \in E\} = \nu(E) \text{ for every measurable } E \subset S^1$$

It is also ergodic: if $f^{-1}(E) = E$ then $\nu(E) = 0$ or $1$.

Birkhoff ergodic theorem: for every $\varphi \in L^1(S^1)$ and $\nu$-a.e. $z \in S^1$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^kz) = \int_{S^1} \varphi(y) \, d\nu(y)$$
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**LAW OF LARGE NUMBERS**
A Bernoulli process

Lebesgue measure on the circle passes to a measure $\mu$ on $\Sigma_2^+ = \{0, 1\}^\mathbb{N}$

- $\mu(E) = \nu(\pi(E))$, where $\pi: \Sigma_2^+ \to S^1$, $x \mapsto \exp(\pi i \sum_{k=0}^{\infty} x_k 2^{-k})$

Define $\varphi: \Sigma_2^+ \to \mathbb{R}$ by $\varphi(x) = x_0$. This gives a sequence of random variables on $(\Sigma_2^+, \mu)$ by $X_n = \varphi(f^n x)$. 
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- Central limit theorem: $\frac{1}{\sqrt{n}} \sum_{k=1}^n (X_k - \mathbb{E}X)$ converges to Gaussian
- Large deviations: Estimates on $P(|\frac{1}{n} \sum_{k=1}^n X_k - \mathbb{E}X| > \delta)$
- Law of the iterated logarithm: $\frac{\sum_{k=1}^n (X_k - \mathbb{E}X)}{\sqrt{n \log \log n}}$ converges to zero in probability but not almost surely
- ...and so on...
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This works for any “nice enough” $\varphi$, and all this happens even though the dynamical system is deterministic.
Abundance of invariant measures

**Idea** Deal with chaotic behaviour by treating the observations \( \varphi \circ f^k \) as random variables.

Requires an invariant measure \( \mu \), and \( \Sigma_2^+ \) has many such measures.

- Bernoulli measures – weighted coin flips
- Periodic orbit measures – atomic
- Everything in between: Markov measures, Gibbs measures, etc.

Some have good statistical properties, some don’t. **Which are natural?**

Look for an absolutely continuous invariant measure (acim).
Expansion and contraction

Can we deal with the logistic map $f_\lambda(x) = \lambda x(1 - x)$ this way?

- Find acim $\mu$, treat $X_n = \phi(f^k x)$ as a stochastic process.
Expansion and contraction

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- Find acim $\mu$, treat $X_n = \varphi(f^k x)$ as a stochastic process.

Doubling map has uniform expansion: $d(fx, fy) = 2d(x, y)$ if $x \sim y$
- Destroys correlations and yields stochastic behaviour

Logistic map has both expansion and contraction:
- $d(fx, fy) < d(x, y)$ if $x, y$ near critical point
- $d(fx, fy) > d(x, y)$ if away from critical point

How much time does a typical orbit spend near critical point?
Typical orbits for logistic map

Consider logistic map \( f(x) = 4x(1 - x) \). "Typical" means w.r.t. Lebesgue, but now Lebesgue measure is not invariant.

- **Fact:** \( d\mu = \pi^{-1}(x(1 - x))^{-1/2} \, dx \) is an ergodic invariant measure
- **Claim:** \( \exists \chi > 0 \) such that typical points \( x \) have \( |(f^n)'(x)| \approx e^{\chi n} \)
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\[
\frac{1}{n} \log |(f^n)'(x)| = \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(f^k x)|
\]  
  (chain rule)

\[
\text{Leb-a.e. } \int_0^1 \log |f'(y)| \, d\mu(y)
\]  
  \[
= \frac{1}{\pi} \int_0^1 \log |4 - 8y| \, dy
\]  
  \[
= \log 2
\]  
  (definition of $f$, $\mu$)  
  (wizardry)
Dependence on parameter value

The parameter $\lambda$ in $f_\lambda(x) = \lambda x(1 - x)$ ranges from 0 to 4.

- When $\lambda = 4$, expansion beats contraction for typical orbits.
- For $0 \leq \lambda \leq 3$, there is an attracting fixed point ( contraction wins). No stochastic behaviour in this case.

$$x = f_\lambda(x) = \lambda x(1 - x) \iff x = 0, 1 - \frac{1}{\lambda}$$

$$f'_\lambda(x) = \lambda - 2\lambda x = \lambda, 2 - \lambda$$
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What happens for $3 < \lambda < 4$? Which is dominant, expansion or contraction?
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Classification of behaviour

At least two types of behaviour:

1. Attracting periodic orbit: $f^p(x) = x$ and $f^n(y) \to \mathcal{O}(x)$ for Leb-a.e. $y$

2. Absolutely continuous invariant measure: $\mu \ll \text{Leb}, \mu \circ f^{-1} = \mu$, and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k y) = \int \varphi(x) \, d\mu(x)$$

for Leb-a.e. $y$ and every $\varphi \in C([0, 1])$

$S = \{ \lambda \in [3, 4] \mid \text{periodic attractor} \}$  \text{(stable behaviour)}

$U = \{ \lambda \in [3, 4] \mid \text{acim} \}$  \text{(unstable behaviour)}
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- \( S \) is open and dense... complement is a Cantor set
- \( U \) has positive Lebesgue measure despite being nowhere dense
Bifurcations

- $\lambda < 3$: one attracting fixed point, no periodic orbits
- $3 < \lambda < 3 + \epsilon$: fixed point is repelling, period-2 orbit is attracting

There is a **bifurcation** at $\lambda = 3$ – qualitative behaviour changes

Another bifurcation happens at $\lambda \approx 3.45$:
- Period-2 orbit becomes unstable
- A stable period-4 orbit is created

Sometime before $\lambda \approx 3.56$ the period-4 orbit becomes unstable and spawns a period-8 orbit...
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Period doubling cascades and universality

At $\lambda_n$, period $2^n$ orbit becomes unstable, period $2^{n+1}$ orbit is born: this is a **period doubling cascade**

$$\lambda_n \rightarrow \lambda_\infty \approx 3.569946 \ldots$$
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$$\lambda_n \rightarrow \lambda_\infty \approx 3.569946 \ldots$$

It turns out that $\lambda_\infty - \lambda_n \approx C\delta^n$, where $\delta \approx 1/4.6692 \ldots$ is the Feigenbaum constant.

**Universality**

This applies to a very large class of one-parameter families $f_\lambda$, not just the logistic maps.
Windows of stability

Contraction beats expansion for $\lambda < \lambda_{\infty}$.

- What happens for $\lambda > \lambda_{\infty}$?

Sometimes expansion wins (there is an acip and chaos), but there are windows of stability where $f_\lambda$ has an attracting periodic orbit.

These windows of stability are dense in $[0, 4]$.
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$$\lambda = 3.835$$
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Sometimes expansion wins (there is an acip and chaos), but there are windows of stability where $f_\lambda$ has an attracting periodic orbit.

\[ f^3(x) = 3.815 \]

These windows of stability are dense in $[0, 4]$

**Theorem:** If there is a period-3 orbit then there are orbits of all periods.
Windows of stability

Contraction beats expansion for $\lambda < \lambda_\infty$.

- What happens for $\lambda > \lambda_\infty$?

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More on windows of stability

Periodic orbits appear in an order given by the Sharkovsky ordering:

\[ 1 \prec 2 \prec 4 \prec 8 \prec 16 \prec \cdots \]
\[ \cdots \]
\[ \cdots \prec 7 \cdot 2^n \prec 5 \cdot 2^n \prec 3 \cdot 2^n \]
\[ \cdots \]
\[ \cdots \prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2 \]
\[ \cdots \prec 7 \prec 5 \prec 3 \]

Each window of stability has its own period doubling cascade. *Self-similarity – a fractal sort of behaviour*

Universality constants are same as before.
Types of chaotic behaviour

Uniform expansion (*doubling map*):
- Phase space expanded at every point
- Along an orbit, expansion at every time
- Stable under perturbations

Non-uniform expansion (*logistic map*):
- Some expansion, some contraction
- Along an orbit, contraction may occur but expansion wins asymptotically
- Very sensitive to perturbations

Higher dimensional (*Lorenz equations*):
- Some directions expand and others contract
- Expansion and contraction may be uniform or non-uniform
Higher dimensions

Mechanism for chaos is **stretching** and **folding** of the phase space.

Formally, given a diffeomorphism $f : M \to M$ of a smooth Riemannian manifold $M$, need a splitting of the tangent bundle:

$$T_x M = E^u(x) \oplus E^s(x)$$

- Invariance: $Df_x E^u(x) = E^u(f(x))$ and $Df_x E^s(x) = E^s(f(x))$
- Expansion in $E^u(x)$ and contraction in $E^s(x)$

**Key step:** Integrate $E^{s,u}$ to stable and unstable manifolds $W^{s,u} \subset M$. 