Non-uniform specification properties and large deviations

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Setting: $X$ a shift space on a finite alphabet (generalises naturally)

Theorem (Known results)

Suppose $X$ has specification. Then

1. bounded distortion $\Rightarrow$ unique equilibrium state.
2. A large deviations principle holds for every Gibbs measure.

Goal: Same results with non-uniform versions of above properties

Key idea:

- $\mathcal{L}$ the language of $X$ (space of finite orbit segments)
- Only require properties for $\mathcal{G} \subset \mathcal{L}$
- Get results if $\mathcal{G}$ is “big enough”
Specification for shift spaces

**Shift space:** closed, shift-invariant set $X \subset \mathcal{A}^\mathbb{N}$

- $\mathcal{A} = \{1, \ldots, p\}$ a finite alphabet

Every finite word $w \in \mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ determines a cylinder

$$[w] = \{ x \in X \mid x_1 \ldots x_n = w \} \quad (n = |w|)$$

The **language** of $X$ is $\mathcal{L} = \{ w \in \mathcal{A}^* \mid [w] \neq \emptyset \}$.

$X$ is **topologically transitive** iff

- for all $u, v \in \mathcal{L}$ there exists $w \in \mathcal{L}$ such that $uwv \in \mathcal{L}$

$X$ has **specification** if

- there exists $\tau \in \mathbb{N}$ such that $w$ can be chosen with $|w| \leq \tau$, independently of the length of $u, v$
Pressure and equilibrium states

Topological pressure of $\varphi: X \to \mathbb{R}$ is

$$P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{w \in L_n} e^{\varphi_n(w)} \right),$$

where $L_n = \{ w \in L \mid |w| = n \}$ and $\varphi_n(w) = \sup_{x \in [w]} S_n \varphi(x)$.

$S_n \varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1} x)$

Variational principle: $P(\varphi) = \sup\{ h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_\sigma(X) \}$

- $\mathcal{M}_\sigma(X) = \{ \sigma\text{-invariant probability measures on } X \}$

A measure achieving the supremum is an equilibrium state.
Unique equilibrium states

φ has **bounded distortions** if there exists $V \in \mathbb{R}$ such that

$$|S_n\varphi(x) - S_n\varphi(y)| \leq V \text{ for all } w \in \mathcal{L}, \ x, y \in [w] \quad (n = |w|)$$

$\mu \in \mathcal{M}_\sigma(X)$ is **Gibbs** if there are $K, K' > 0$ such that

$$K \leq \frac{\mu[w]}{e^{-nP(\varphi) + S_n\varphi(x)}} \leq K'$$

for all $w \in \mathcal{L}$, $n = |w|$, $x \in [w]$. 

**Theorem (Bowen, 1974)**

*If $X$ has specification and $\varphi$ has bounded distortions, then $\varphi$ has a unique equilibrium state $\mu$, and $\mu$ has the Gibbs property.*
Empirical measures

\[ \mathcal{M}(X) = \{ \text{Borel probability measures on } X \} \]

Given \( x \in X \) and \( n \in \mathbb{N} \), get empirical measure

\[ E_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\sigma^k x} \]

\[ E_n(x)(\varphi) = S_n \varphi(x) \]

Recall \( E_n(x) \to m \) for \( m \)-a.e. \( x \) if \( m \) ergodic

Large deviations studies rate of decay of \( m\{ x \mid E_n(x) \in U \} \) for sets \( U \subset \mathcal{M}(X) \) not containing \( m \).
Large deviations

$X$ satisfies a large deviations principle with reference measure $m$ and rate function $q : \mathcal{M}(X) \rightarrow [-\infty, 0]$ if

\[
U \subset \mathcal{M}_\sigma(X) \text{ open } \Rightarrow \liminf_{n \to \infty} \frac{1}{n} \log m\{x | \mathcal{E}_n(x) \in U\} \geq \sup_{\mu \in U} q(\mu)
\]

\[
F \subset \mathcal{M}_\sigma(X) \text{ closed } \Rightarrow \limsup_{n \to \infty} \frac{1}{n} \log m\{x | \mathcal{E}_n(x) \in F\} \leq \sup_{\mu \in F} q(\mu)
\]

Theorem

If $X$ has specification and $m$ is Gibbs for $\varphi$, then $X$ satisfies a large deviations principle with reference measure $m$ and rate function

\[
q(\mu) = \begin{cases} 
  h(\mu) + \int \varphi \, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\
  -\infty & \mu \notin \mathcal{M}_\sigma(X)
\end{cases}
\]
For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta: [0, 1] \to [0, 1], \quad x \mapsto \beta x \pmod{1}$$

$$1_\beta = a_1a_2\cdots, \text{ where } 1 = \sum_{n=1}^{\infty} a_n\beta^{-n}$$

**Fact:**

$$x \in \Sigma_\beta \iff \sigma^n x \leq 1_\beta \text{ for all } n$$

$$\iff x \text{ labels a walk starting at } B \text{ on this graph:}$$

(Here $1_\beta = 2100201\ldots$)
Properties of $\beta$-shifts

$\Sigma_\beta$ has specification iff $1_\beta$ does not contain arbitrarily long sequences of 0s.

Schmeling (1997): For Leb-a.e. $\beta$, $\Sigma_\beta$ does not have specification

Hofbauer (1979): $\Sigma_\beta$ has a unique measure of maximal entropy

Walters (1978): Every Lipschitz potential has a unique eq. state

Equilibrium state is not Gibbs – so what about large deviations? And what about more general bounded distortion potentials?
Coded systems

Given $\beta > 1$, $\alpha \in (0, 1)$, consider coding space for

$$f_{\alpha, \beta} : x \mapsto \alpha + \beta x \pmod{1}.$$ 

Can be presented on a countable graph, but more complicated. Similarly with any piecewise expanding interval map.

General class of coded systems: two equivalent characterisations

- Can be presented on a countable graph (finitely many labels)
- Countable set $G$ of generating words $w^j$ that can be freely concatenated: $\mathcal{L} = \overline{G^*} = \{ \text{subwords of } w^{j_1} \cdots w^{j_n} \}$

Given $S \subset \mathbb{N}$, consider the words $w^n = 0^n1$ for each $n \in S$.

- $\Sigma_S$ is the coded system with generators $\{ w^n \mid n \in S \}$. $\Sigma_S$ has specification iff $S$ is syndetic (bounded gaps)
Potentials with unbounded distortion

Manneville–Pomeau / Hofbauer-type potentials on $\Sigma_\beta$:

- $x \in \Sigma_\beta$: $k(x) =$ number of initial 0s in $x$
- $\varphi(x) = a_{k(x)}$ where $a_n \to 0$ as $n \to \infty$
- If $|\sum a_n| = \infty$, then $\varphi$ has unbounded distortion and $P(t\varphi)$ can exhibit phase transitions
- Arises from $f(x) = x + \gamma x^{1+\varepsilon} \pmod{1}$ for $\gamma > 0$, $\varphi = -\log f'$

Grid potentials:

- $\varphi(x) = \psi(x) + a_{k(x)}$, where $\psi$ has bounded distortion and $2^{-k(x)}$ is distance from $x$ to some subshift $Y \subset X$
Collections of words

\( \mathcal{X} \) a shift space, \( \mathcal{L} \) its language, \( \mathcal{D} \subset \mathcal{L} \)

**Pressure of \( \varphi \) on \( \mathcal{D} \).** Let \( \mathcal{D}_n = \{w \in \mathcal{D} \mid |w| = n\} \), then

\[
P(\mathcal{D}, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \mathcal{D}_n} e^{\varphi_n(w)} \quad h(\mathcal{D}) = P(\mathcal{D}, 0)
\]

\( \mathcal{D} \) has specification if there exists \( \tau \in \mathbb{N} \) such that for all \( w^1, \ldots, w^k \in \mathcal{D} \), there exist \( v^1, \ldots, v^{k-1} \in \mathcal{L} \) with \( |v^j| \leq \tau \) such that \( w^1 v^1 w^2 \cdots v^{k-1} w^k \in \mathcal{L} \).

\( \varphi \) has bounded distortion on \( \mathcal{D} \) if there exists \( V \in \mathbb{R} \) such that for all \( w \in \mathcal{D} \), \( n = |w| \), \( x, y \in \lfloor w \rfloor \), we have \( |S_n \varphi(x) - S_n \varphi(y)| \leq V \).

\( \mu \) has the Gibbs property on \( \mathcal{D} \) if there are \( K, K' > 0 \) such that for all \( w \in \mathcal{D} \), \( n = |w| \), \( x \in \lfloor w \rfloor \), we have \( K \leq \frac{\mu[w]}{e^{-nP(\varphi) + S_n \varphi(x)}} \leq K' \).
Decompositions

**Idea:** Unique equilibrium state for $\varphi$ if there is a “large enough” $G \subset \mathcal{L}$ with specification such that $\varphi$ has bounded distortion on $\mathcal{D}$.

What does “large enough” mean?

**Decomposition** of $\mathcal{L}$: sets $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$.

$$G^M = \{uvw \in \mathcal{L} \mid u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u|, |w| \leq M\}$$

**Theorem (C.–Thompson, 2012)**

Suppose $\mathcal{L}$ has a decomposition such that

1. $\varphi$ has bounded distortion on $\mathcal{G}$
2. $G^M$ has specification for every $M$
3. $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\varphi)$

Then $\varphi$ has a unique equilibrium state $\mu$. It is Gibbs on each $G^M$. 
Example: $\beta$-shift

$C^p = \emptyset$

$G = \{\text{words (paths) starting and ending at } B\}$

$C^s = \{\text{words (paths) starting at } B \text{ and never returning}\}$

$L = C^p GC^s$

$G^M$ corresponds to paths ending in first $M$ vertices, so $G^M$
has specification for each $M$

$h(C) = 0$, where $C = C^p \cup C^s$
Hölder potentials

To get unique equilibrium state for \( \varphi \), need \( P(C, \varphi) < P(\varphi) \).

Suppose we know that \( h(C) + \sup_{x \in X} (\lim \frac{1}{n} S_n \varphi(x)) < P(\varphi) \).
Then get \( P(C, \varphi) < P(\varphi) \).

Equivalent conditions:

- \( \sup_{x} \lim \frac{1}{n} S_n \varphi(x) < P(\varphi) - h(C) \)
- \( \exists n \) such that \( \sup_{x} \frac{1}{n} S_n \varphi(x) < P(\varphi) - h(C) \)
- Every equilibrium state for \( \varphi \) has \( h(\mu) > h(C) \)

Theorem (C.–Thompson, 2012)

When \( X \) is a \( \beta \)-shift, every Hölder continuous potential satisfies the above condition. In particular, it has a unique equilibrium state \( \mu \), and \( \mu \) is Gibbs on each \( G^M \).
Decompositions for coded systems

Let $X$ be a coded shift: two natural decompositions of $\mathcal{L}$.

In terms of countable graph presentation
- Fix a finite subset $F$ of the graph
- $C^p = \text{paths starting outside } F \text{ and entering it only on the last step, or never}$
- $G = \text{paths starting and ending in } F$
- $C^s = \text{paths starting in } F \text{ and never returning}$

In terms of generators
- $G \subset A^*$ a set of generators
- $G = G^* = \{w^1 \cdots w^n \mid w^j \in G\}$
- $C^p = \text{suffixes of generators}$
- $C^s = \text{prefixes of generators}$
Let $f$ be a piecewise expanding interval map, $X$ the coding space

- Graph presentation gives decomposition of $L$
- $h(C) > 0$, but can be made arbitrarily small by taking $F$ large

**Definition**

The entropy of obstructions to specification is

$$h_{\text{spec}}(X) = \inf \{ h(C^p \cup C^s) \mid \text{there exists } G \text{ such that} \ L = C^p G C^s \text{ and every } G^M \text{ has specification} \}$$

Unique equilibrium state for $\varphi$, Gibbs on each $G^M$, if any (all) of

- $\sup_x \overline{\lim} \frac{1}{n} S_n \varphi(x) < P(\varphi) - h_{\text{spec}}$
- $\exists n$ such that $\sup_x \frac{1}{n} S_n \varphi(x) < P(\varphi) - h_{\text{spec}}$
- Every equilibrium state for $\varphi$ has $h(\mu) > h_{\text{spec}}$
Positive entropy equilibrium states

For $S$-gap shifts the natural decomposition from generators gives
\begin{itemize}
  \item $C^p = \{0^n 1 \mid n \in \mathbb{N}\}$
  \item $G = \{0^{n_1} 1 \cdots 0^{n_k} 1 \mid n_j \in S\}
  \item $C^s = \{0^n \mid n \in \mathbb{N}\}$
\end{itemize}

So $h_{\text{spec}}^\perp = 0$, as with other examples.

For $S$-gap shifts, every Hölder potential has $P(\varphi) > \sup \lim \frac{1}{n} S_n \varphi$.

This gives same set of results as for $\beta$-shifts: unique equilibrium state, Gibbs on $G^M$.

Open questions: What about piecewise expanding interval maps? Are there coded systems with $h_{\text{spec}}^\perp = 0$ for which some Hölder potentials have zero entropy equilibrium states?
Unbounded distortion

$X$ a $\beta$-shift, $\varphi(x) = \psi(x) + a_k(x)$

$G =$ paths starting at $B$ and ending at the next vertex from $B$

$C^s =$ paths starting at the second vertex and never returning to $B$, and paths starting at $B$ and never getting to the next vertex

- $G^M$ has specification for each $M$
- $\varphi$ has bounded distortion on $G$
- $P(C^s, \varphi) < P(\varphi)$ whenever $\varphi(0) < P(\varphi)$
Manneville–Pomeau for $\beta$-transformations

**Conclusion:** If $P(\varphi) > \varphi(0)$ then $\varphi$ has a unique equilibrium state.

**Corollary:** Let $f(x) = x + \gamma x^{1+\varepsilon} \pmod{1}$ and $\varphi(x) = -\log f'(x)$, where $\gamma > 0$ and $0 < \varepsilon < 1$. Then $t\varphi$ has a unique equilibrium state for every $t < 1$.

Can get similar results with grid potentials if $X$ has specification.
Large deviations results have been obtained for $\beta$-shift and other systems by using statistical specification properties.

- Pfister, Sullivan (2005)
- Yamamoto (2009)
- Varandas (2012)

All reflect idea that the gluing procedure can be weakened in a way that does not interfere too much with Birkhoff averages.
Given any $v \in \mathcal{L}$, can transform $v$ into a word $u \in \mathcal{G}$ by making a single change. *(Change last non-zero symbol to 0).*

Thus given any $v, w \in \mathcal{L}$, the word $vw$ may not be in $\mathcal{L}$, but can be transformed into a word in $\mathcal{L}$ by making a single change.

General method for getting a word that concatenates statistical properties of $v$ and $w$, as long as $\frac{\text{number of changes}}{\text{length of word}} \to 0$. 
Edit metric

Goal: Define a metric on $A^*$ (set of all finite words) that controls how much Birkhoff sums can vary.

An edit of a word $w$ is any of the following:

- **Substitution:** $w = uav \mapsto w' = ubv$  
  $u, v \in A^*, \ a, b \in A$

- **Insertion:** $w = uv \mapsto w' = ubv$  
  $u, v \in A^*, \ b \in A$

- **Deletion:** $w = uav \mapsto w' = uv$  
  $u, v \in A^*, \ a \in A$

$\hat{d}(v, w) = \text{minimum number of edits required to go from } v \text{ to } w.$

**Key property:** Let $D$ be a metric inducing the weak* topology on $\mathcal{M}(X)$. Then for every $\eta > 0$ there is $\delta > 0$ such that if $\frac{\hat{d}(v, w)}{|v|} < \delta$, then $D(\mathcal{E}_{|v|}(x), \mathcal{E}_{|w|}(y)) < \eta$ for all $x \in [v]$ and $y \in [w]$. 
Edit approachability

mistake function: a non-increasing sub-linear function $g: \mathbb{N} \to \mathbb{N}$.
\[
\left( \frac{g(n)}{n} \to 0 \right)
\]

$L$ is edit approachable by $G \subset L$ if there exists a mistake function $g$ such that for every $v \in L$, there is $w \in G$ with $\hat{d}(v, w) < g(|v|)$.

Equivalently, $L = \bigcup_{w \in G} B_{\hat{d}}(w, g(|w|))$.

Examples: For both the $\beta$-shifts and the $S$-gap shifts, $L$ is edit approachable by the natural choice of $G$. 
**Large deviations**

Theorem (C.–Thompson–Yamamoto, 2013)

Let $X$ be a shift space on a finite alphabet, $\mathcal{L}$ its language. Suppose

1. $\mathcal{L}$ is edit approachable by $\mathcal{G}$,
2. $\mathcal{G}$ has specification (with good concatenations),
3. $m \in \mathcal{M}(X)$ is Gibbs for $\varphi$ on $\mathcal{G}$.

Then $X$ satisfies a LDP with reference measure $m$ and rate $f'n$

$$q(\mu) = \begin{cases} h(\mu) + \int \varphi \, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\ -\infty & \mu \notin \mathcal{M}_\sigma(X) \end{cases}$$

In particular, every Hölder continuous $\varphi$ on a $\beta$-shift or $S$-gap shift.
Key tool in proof

The bulk of the proof is in the following proposition.

$X$ a shift space, $\mathcal{L}$ edit approachable by $\mathcal{G}$ with specification

Then $\exists$ an increasing sequence $X_n \subset X$ of subshifts s.t.

1. Each $X_n$ has specification
2. If $m$ is Gibbs on $\mathcal{G}$, then it is Gibbs on every $\mathcal{L}(X_n)$
3. For every $\mu \in \mathcal{M}_\sigma(X)$ there are subshifts $Y_n \subset X_n$ s.t.
   $\mathcal{M}_\sigma(Y_n) \to \{\mu\}$ and $\lim h(Y_n) \geq h(\mu)$

In particular, ergodic measures are entropy-dense in $\mathcal{M}_\sigma(X)$
One moral of the story: Many good consequences of specification (and other properties) can still be obtained as long as properties hold on a “large enough” set of words (orbit segments).

“Large enough” means the ability to get from $\mathcal{L}$ to $\mathcal{G}$ with some “small” tinkering, where meaning of “small” depends on context.

- Unique equilibrium state: only need to remove a prefix and a suffix from the word in $\mathcal{L}$, and these come from “small” lists
- Large deviations: only need to make a small number of edits
- Hölder $\Rightarrow$ only positive entropy equilibrium states: ???